Physical Interpretation for the Generalized Parton Distributions
$H(x, 0, -\Delta^2_\perp)$ and $E(x, 0, -\Delta^2_\perp)$

or: What DVCS has to do with the distribution of partons in the transverse plane

Matthias Burkardt

burkardt@nmsu.edu

New Mexico State University
Las Cruces, NM, 88003, U.S.A.
Outline

- Motivation
- Deep-inelastic scattering (DIS)
- Generalized parton distributions (GPDs)
- Probabilistic interpretation of GPDs
  - $H(x, 0, -\Delta^2_\perp) \rightarrow q(x, b_\perp)$
  - $\tilde{H}(x, 0, -\Delta^2_\perp) \rightarrow \Delta q(x, b_\perp)$
  - $E(x, 0, -\Delta^2_\perp) \rightarrow \perp$ displacement of quark distributions
- Summary
Motivation:

(see talk by A. Belitski)


\[
\langle J_q \rangle = \frac{1}{2} \int_0^1 dx \, x [H_q(x, 0, 0) + E_q(x, 0, 0)]
\]

\[\text{DVCS} \iff \text{GPDs} \iff \vec{J}_q\]

\[\rightarrow \text{lots of works about GPDs}\]

\[\rightarrow \text{GPDs very useful tool linking variety of observables}\]

- But: what other “physical information” about the nucleon can we obtain by measuring/calculating GPDs?
DIS

light-cone coordinates:

\[ x^+ = \frac{x^0 + x^3}{\sqrt{2}} \]
\[ x^- = \frac{x^0 - x^3}{\sqrt{2}} \]

DIS related to correlations along light–cone

\[ q(x_{Bj}) = \int \frac{dx^-}{2\pi} \langle P|\overline{q}(0^-, 0_\perp)\gamma^+ q(x^-, 0_\perp)|P\rangle e^{ix^-x_{Bj}P^+} \]

No information about transverse position of partons!

Physical Interpretation for the Generalized Parton Distributions \( H(x, 0, -\Delta^2_\perp) \) and \( E(x, 0, -\Delta^2_\perp) \) – p.4/55
Deeply Virtual Compton Scattering (DVCS)

\[ T^{\mu\nu} = i \int d^4z \ e^{iz\cdot q} \left\langle p' \left| T J^\mu \left( -\frac{z}{2} \right) J^\nu \left( \frac{z}{2} \right) \right| p \right\rangle \]

\[ B_{\parallel} \rightarrow \frac{g_{\perp}^{\mu\nu}}{2} \int_{-1}^{1} dx \left( \frac{1}{x - \xi + i\epsilon} + \frac{1}{x + \xi - i\epsilon} \right) H(x, \xi, \Delta^2) \bar{u}(p') \gamma^+ u(p) + \ldots \]

\[ \bar{q} = (q + q')/2 \quad \Delta = p' - p \quad x_{B_{\parallel}} \equiv -q^2/2p \cdot q = 2\xi(1 + \xi) \]
Generalized Parton Distributions (GPDs)

\[
\int \frac{dx^-}{2\pi} e^{ix^--p^+x} \left\langle p' \left| q \left( -\frac{x^-}{2} \right) \gamma^+ q \left( \frac{x^-}{2} \right) \right| p \right\rangle = H(x, \xi, \Delta^2) \tilde{u}(p') \gamma^+ u(p) + E(x, \xi, \Delta^2) \tilde{u}(p') \frac{i\sigma^{+\nu} \Delta_\nu}{2M} u(p),
\]

\[
\int \frac{dx^-}{2\pi} e^{ix^--p^+x} \left\langle p' \left| \bar{q} \left( -\frac{x^-}{2} \right) \gamma^+ \gamma_5 q \left( \frac{x^-}{2} \right) \right| p \right\rangle = \tilde{H}(x, \xi, \Delta^2) \bar{u}(p') \gamma^+ \gamma_5 u(p) + \tilde{\bar{E}}(x, \xi, \Delta^2) \bar{u}(p') \frac{\gamma_5 \Delta^+}{2M} u(p).
\]
Parton Interpretation

- $x$ is mean long. momentum fraction carried by active quark
- $\Delta^+ = 2\xi \bar{p}^+$, i.e. $\xi$ is the long. momentum transfer on quark/target
- $\Delta_\perp$ is perp. momentum transfer on target
- In general no probabilistic interpretation since initial and final state not the same
- instead: interpretation as transition amplitude
- GPDs tell how much quarks with momentum fraction $x$ (in IMF) contribute to form factor
- depend on 3 variables since 1.) $x$ is ‘measured’; 2.) LC correlation singles out $z$-direction
What is Physics of GPDs?

- **Definition of GPDs** resembles that of form factors

\[
\langle p' \big| \hat{O} \big| p \rangle = H(x, \xi, \Delta^2) \bar{u}(p')\gamma^+ u(p) + E(x, \xi, \Delta^2) \bar{u}(p') \frac{i\sigma^{+\nu} \Delta_\nu}{2M} u(p)
\]

with \( \hat{O} \equiv \int \frac{dx^-}{2\pi} e^{ix^-\vec{p}^+} x \vec{q} \left( -\frac{x^-}{2} \right) \gamma^+ q \left( \frac{x^-}{2} \right) \)

- relation between PDFs and GPDs similar to relation between a charge and a form factor
- If form factors can be interpreted as Fourier transforms of charge distributions in position space, what is the analogous physical interpretation for GPDs?
# Form Factors vs. GPDs

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<td>$F(t)$</td>
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<td>$q(x)$</td>
<td>$H(x, \xi, t)$</td>
<td>?</td>
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Physical Interpretation for the Generalized Parton Distributions $H(x, 0, -\Delta^2_\perp)$ and $E(x, 0, -\Delta^2_\perp)$ – p.9/55
Form Factors vs. GPDs

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$q(x, b_\perp) = \text{impact parameter dependent PDF}$
Impact parameter dependent PDF

- define state that is localized in $\perp$ position:

$$|p^+, R_\perp = 0_\perp, \lambda\rangle = N \int d^2 p_\perp |p^+, p_\perp, \lambda\rangle$$

Note: $\perp$ boosts in IMF form Galilean subgroup

$\Rightarrow$ this state has $R_\perp \equiv \sum_i x_i b_{\perp,i} = 0_\perp$

(cf.: working in CM frame in nonrel. physics)

- define impact parameter dependent PDF

$$q(x, b_\perp) = \int \frac{dx^-}{4\pi} \left< p^+, 0_\perp \middle| \bar{\psi}(-\frac{x^-}{2}, b_\perp) \gamma^+ \psi(\frac{x^-}{2}, b_\perp) \middle| p^+, 0_\perp \right> e^{ixp^+x^-}$$
use translational invariance to relate to same matrix element that appears in def. of GPDs

\[
q(x, b_\perp) \equiv \int dx^- \langle p^+, R_\perp = 0_\perp \mid \bar{\psi}\left(-\frac{x^-}{2}, b_\perp\right)\gamma^+ \psi\left(\frac{x^-}{2}, b_\perp\right) \mid p^+, R_\perp = 0_\perp \rangle e^{ixp^+x^-}
\]

\[
= |N|^2 \int d^2 p_\perp \int d^2 p'_\perp \int dx^- \langle p^+, p'_\perp \mid \bar{\psi}\left(-\frac{x^-}{2}, b_\perp\right)\gamma^+ \psi\left(\frac{x^-}{2}, b_\perp\right) \mid p^+, p'_\perp \rangle e^{ixp^+x^-}
\]
use translational invariance to relate to same matrix element that appears in def. of GPDs

\[ q(x, b_\perp) \equiv \int dx^- \langle p^+, R_\perp = 0_\perp | \bar{\psi}(\frac{-x^-}{2}, b_\perp) \gamma^+ \psi(\frac{x^-}{2}, b_\perp) | p^+, R_\perp = 0_\perp \rangle e^{i x p^+ x^-} \]

\[ = |\mathcal{N}|^2 \int d^2 p_\perp \int d^2 p'_\perp \int dx^- \langle p^+, p'_\perp | \bar{\psi}(\frac{-x^-}{2}, b_\perp) \gamma^+ \psi(\frac{x^-}{2}, b_\perp) | p^+, p_\perp \rangle e^{i x p^+ x^-} \]

\[ = |\mathcal{N}|^2 \int d^2 p_\perp \int d^2 p'_\perp \int dx^- \langle p^+, p'_\perp | \bar{\psi}(\frac{-x^-}{2}, 0_\perp) \gamma^+ \psi(\frac{x^-}{2}, 0_\perp) | p^+, p_\perp \rangle e^{i x p^+ x^-} \times e^{i b_\perp \cdot (p_\perp - p'_\perp)} \]
use translational invariance to relate to same matrix element that appears in def. of GPDs

\[ q(x, b_{\perp}) \equiv \int dx^- \langle p^+, R_\perp = 0_{\perp} | \bar{\psi}(-\frac{x^-}{2}, b_{\perp}) \gamma^+ \psi(\frac{x^-}{2}, b_{\perp}) | p^+, R_\perp = 0_{\perp} \rangle e^{i x p^+ x^-} \]

\[ = |N|^2 \int d^2 p_\perp \int d^2 p'_\perp \int dx^- \langle p^+, p'_\perp | \bar{\psi}(-\frac{x^-}{2}, b_{\perp}) \gamma^+ \psi(\frac{x^-}{2}, b_{\perp}) | p^+, p_\perp \rangle e^{i x p^+ x^-} \]

\[ = |N|^2 \int d^2 p_\perp \int d^2 p'_\perp \int dx^- \langle p^+, p'_\perp | \bar{\psi}(-\frac{x^-}{2}, 0_{\perp}) \gamma^+ \psi(\frac{x^-}{2}, 0_{\perp}) | p^+, p_\perp \rangle e^{i x p^+ x^-} \]

\[ \times e^{i b_{\perp} \cdot (p_{\perp} - p'_{\perp})} \]

\[ = |N|^2 \int d^2 p_\perp \int d^2 p'_\perp H \left( x, 0, -(p'_\perp - p_{\perp})^2 \right) e^{i b_{\perp} \cdot (p_{\perp} - p'_{\perp})} \]

\[ \leftrightarrow \]

\[ q(x, b_{\perp}) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} H(x, -\Delta^2_{\perp}) e^{-i b_{\perp} \cdot \Delta_\perp} \]
\( q(x, b_{\perp}) \) has physical interpretation of a **density**

\[
q(x, b_{\perp}) \geq 0 \quad \text{for} \quad x > 0 \\
q(x, b_{\perp}) \leq 0 \quad \text{for} \quad x < 0
\]
Discussion

- GPDs allow simultaneous determination of longitudinal momentum and transverse position of partons

\[ q(x, b_\perp) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} H(x, 0, -\Delta_\perp^2) e^{-ib_\perp \cdot \Delta_\perp} \]

- \( q(x, b_\perp) \) has interpretation as density (positivity constraints!)

\[ q(x, b_\perp) \sim \langle p^+, 0_\perp | b^\dagger(xp^+, b_\perp)b(xp^+, b_\perp) | p^+, 0_\perp \rangle \]
\[ = \left| b(xp^+, b_\perp) | p^+, 0_\perp \rangle \right|^2 \geq 0 \]

\( \iff \) positivity constraint on models
• Nonrelativistically such a result not surprising! Absence of relativistic corrections to identification $H(x, 0, -\Delta_\perp^2) \xrightarrow{FT} q(x, b_\perp)$ due to Galilean subgroup in IMF

• $b_\perp$ distribution measured w.r.t. $R_{CM}^{CM} \equiv \sum_i x_i r_i, \perp \leftrightarrow$ width of the $b_\perp$ distribution should go to zero as $x \to 1$, since the active quark becomes the $\perp$ center of momentum in that limit! $\leftrightarrow H(x, t)$ must become $t$-indep. as $x \to 1$.

• very similar results for impact parameter dependent polarized quark distributions

$$\Delta q(x, b_\perp) = \int \frac{d^2 \Delta_\perp}{(2\pi)^2} \tilde{H}(x, 0, -\Delta_\perp^2) e^{-ib_\perp \cdot \Delta_\perp}$$
Use intuition about nucleon structure in position space to make predictions for GPDs:

- **large** \( x \): quarks from **localized** valence ‘core’,
- **small** \( x \): contributions from **larger** ‘meson cloud’

\( x \) decreases:
- **small** \( x \), expect transverse size to increase
- **model:**
  \[
  H_q(x, 0, -\Delta_\perp^2) = q(x)e^{-a\Delta_\perp^2(1-x)\ln\frac{1}{x}}.
  \]

(consistent with Regge behavior at small \( x \) and quark counting for \( x \to 1 \))
Other topics

- The physics of $E(x, 0, t)$
- QCD evolution
- extrapolating to $\xi = 0$
Summary

- DVCS allows probing GPDS

\[
\int \frac{dx^-}{2\pi} e^{ixp^+x^-} \left\langle p' \left| \bar{\psi} \left( -\frac{x^-}{2} \right) \gamma^+ \psi \left( \frac{x^-}{2} \right) \right| p \right\rangle
\]

- GPDs resemble both PDFs and form factors: defined through matrix elements of light-cone correlation, but \( \Delta \equiv p' - p \neq 0 \).

- \( t \)-dependence of GPDs at \( \xi = 0 \) (only \( \perp \) momentum transfer) \( \Rightarrow \) Fourier transform of impact parameter dependent parton distributions \( q(x, b_\perp) \)

\( \Rightarrow \) knowledge of GPDs for \( \xi = 0 \) allows determining distribution of partons in the \( \perp \) plane

Physical Interpretation for the Generalized Parton Distributions \( H(x, 0, -\Delta_\perp^2) \) and \( E(x, 0, -\Delta_\perp^2) \) - p.20/55
GPDs provide novel information about nonperturbative parton structure of nucleons: 

\[ q(x, b_\perp) = \int \frac{d^2 b_\perp}{(2\pi)^2} \mathcal{H}(x, 0, -\Delta_\perp^2) e^{-ib_\perp \cdot \Delta_\perp} \]

\[ \Delta q(x, b_\perp) = \int \frac{d^2 b_\perp}{(2\pi)^2} \tilde{\mathcal{H}}(x, 0, -\Delta_\perp^2) e^{-ib_\perp \cdot \Delta_\perp} \]

- GPDs provide novel information about nonperturbative parton structure of nucleons:
  - distribution of partons in \( \perp \) plane  
  - \( L_x \sim \langle yp_z - zp_y \rangle \) only lowest \( b_\perp \) moment of that information

- \( q(x, b_\perp), \Delta q(x, b_\perp) \) have probabilistic interpretation, e.g. \( q(x, b_\perp) > 0 \) for \( x > 0 \)

- Universal prediction: large \( x \) partons more localized in \( b_\perp \) than small \( x \) partons
\[ \frac{\Delta_+}{2M} E(x, -\Delta_2) \] describes how the momentum distribution of unpolarized partons in the \( \perp \) plane gets transversely shifted (distorted) if is nucleon polarized in \( \perp \) direction.

The physics of $E(x, 0, -\Delta^2_{\perp})$

So far: only unpolarized (or long. polarized) nucleon

In general, use ($\Delta^+ = 0$)

$$\int \frac{dx^-}{4\pi} e^{ip^+ x^-} \langle P+\Delta, \uparrow \mid \bar{q}(0) \gamma^+ q(x^-) \mid P, \uparrow \rangle = H(x, 0, -\Delta^2_{\perp})$$

$$\int \frac{dx^-}{4\pi} e^{ip^+ x^-} \langle P+\Delta, \uparrow \mid \bar{q}(0) \gamma^+ q(x^-) \mid P, \downarrow \rangle = -\frac{\Delta x - i\Delta y}{2M} E(x, 0, -\Delta^2_{\perp}).$$

Consider nucleon polarized in $x$ direction (in IMF)

$|X\rangle \equiv |p^+, R_{\perp} = 0_{\perp}, \uparrow\rangle + |p^+, R_{\perp} = 0_{\perp}, \downarrow\rangle$.

$\leftrightarrow$ quark distribution for this state

$$q_X(x, b_{\perp}) = q(x, b_{\perp}) - \frac{1}{2M} \frac{\partial}{\partial b_y} \int \frac{d^2\Delta_{\perp}}{(2\pi)^2} E(x, -\Delta^2_{\perp}) e^{-ib_{\perp} \cdot \Delta_{\perp}}$$
gets shifted (distorted) compared to longitudinally polarized nucleons

- mean displacement ($\perp$ flavor dipole moment)

\[
d_{y}^{q} \equiv \int dx \int d^2 b_{\perp} q_{X}(x, b_{\perp}) b_{y} = \frac{1}{2M} \int dx E_{q}(x, 0, 0)
\]

with

\[
k_{u/d} \equiv F_{2}^{u/d}(0) = O(1) \quad \Rightarrow \quad d_{y}^{q} = O(0.1 \text{ fm})
\]

- CM for flavor $q$ shifted relative to CM for whole proton by

\[
\int dx \int d^2 b_{\perp} q_{X}(x, b_{\perp}) x b_{y} = \frac{1}{2M} \int dx x E_{q}(x, 0, 0)
\]
• $q_X(x, b_\perp) \geq 0$

→ positivity constraint for FT of $E(x, 0, -\Delta_{\perp}^2)$:

$$\left| \frac{\nabla b_\perp}{2M} \int \frac{d^2 \Delta_{\perp}}{(2\pi)^2} e^{-ib_\perp \cdot \Delta_{\perp}} E(x, -\Delta_{\perp}^2) \right| < q(x, b_\perp)$$

• polarized GPD with helicity flip $\Rightarrow$

$$\left| \frac{\nabla b_\perp}{2M} \int \frac{d^2 \Delta_{\perp}}{(2\pi)^2} e^{-ib_\perp \cdot \Delta_{\perp}} E(x, -\Delta_{\perp}^2) \right|^2 < |q(x, b_\perp)|^2 - |\Delta q(x, b_\perp)|^2$$
physical origin for $\perp$ distortion

Comparison of a non-rotating sphere that moves in $z$ direction with a sphere that spins at the same time around the $z$ axis and a sphere that spins around the $x$ axis. When the sphere spins around the $x$ axis, the rotation changes the distribution of momenta in the $z$ direction (adds/subtracts to velocity for $y > 0$ and $y < 0$ respectively). For the nucleon the resulting modification of the (unpolarized) momentum distribution is described by $E(x, t)$. 
simple model for $E_q(x, 0, -\Delta^2_\perp)$

- $E_u(x, 0, -\Delta^2_\perp) = \frac{\kappa_u}{2} H_u(x, 0, -\Delta^2_\perp)$
- $E_d(x, 0, -\Delta^2_\perp) = \kappa_d H_d(x, 0, -\Delta^2_\perp)$

with $H_q(x, 0, -\Delta^2_\perp) = q(x) e^{-a \Delta^2_\perp (1-x) \ln \frac{1}{x}}$ and

$\kappa_u = 2\kappa_p + \kappa_n = 1.673 \quad \kappa_d = 2\kappa_n + \kappa_p = -2.033.$

- satisfies $\int dx E_q(x, 0, 0) = \kappa_q$
Physical Interpretation for the Generalized Parton Distributions $H(x, Q, -\Delta_{\perp}^2)$ and $E(x, 0, -\Delta_{\perp}^2)$ - p.28/55
Application: hyperon polarization

model for hyperon polarization in $pp \rightarrow Y + X$
($Y \in \Lambda, \Sigma, \Xi$) at high energy:

- peripheral scattering
- $s\bar{s}$ produced in overlap region, i.e. on “inside track”

$\rightarrow$ if $Y$ deflected to left then $s$ produced on left side of $Y$ (and vice versa)

$\rightarrow$ if $\kappa_s > 0$ then intermediate state has better overlap with final state $Y$ that has spin down (looking into the flight direction)

$\rightarrow$ remarkable prediction: $\vec{P}_Y \sim -\kappa_s^Y \vec{p}_P \times \vec{p}_Y$.
SU(3) analysis for $\kappa_s^B$ yields (assuming $|\kappa_s^p| \ll |\kappa_u^p|, |\kappa_d^p|$)

\[
\begin{align*}
\kappa_s^\Lambda &= \kappa_s^p + \kappa_s^p = 1.79 + \kappa_s^p \\
\kappa_s^\Sigma &= \kappa_s^p + 2\kappa_n^p + \kappa_s^p = -2.03 + \kappa_s^p \\
\kappa_s^\Xi &= 2\kappa_s^p + \kappa_n^p + \kappa_s^p = 1.67 + \kappa_s^p.
\end{align*}
\]

→ expect (polarization $\mathcal{P}$ w.r.t. $\vec{P}_p \times \vec{P}_Y$)

\[
\mathcal{P}_\Lambda < 0 \quad \mathcal{P}_\Sigma > 0 \quad \mathcal{P}_\Xi < 0
\]

→ exp. result:

\[
0 < \mathcal{P}_\Sigma^0 \approx \mathcal{P}_\Sigma^- \approx \mathcal{P}_\Sigma^+ \approx -\mathcal{P}_\Lambda \approx -\mathcal{P}_\Xi^0 \approx -\mathcal{P}_\Xi^-
\]
Figure 1: $P + P(\bar{P}) \rightarrow Y + \bar{Y}$ where the incoming $P$ (from bottom) is deflected to the left during the reaction. The $s\bar{s}$ pair is assumed to be produced roughly in the overlap region, i.e. on the left ‘side’ of the $Y$. 
Figure 2: Schematic view of the transverse distortion of the $s$ quark distribution (in grayscale) in the transverse plane for a transversely polarized hyperon with $\kappa_s^Y > 0$. The view is (from the rest frame) into the direction of motion (i.e. momentum into plane) for a hyperon that moves with a large momentum. In the case of spin down (a), the $s$-quarks get distorted towards the left, while the distortion is to the right for the case of spin up (b).
extrapolating to $\xi = 0$

- bad news: $\xi = 0$ not directly accessible in DVCS since long. momentum transfer necessary to convert virtual $\gamma$ into real $\gamma$

- good news: moments of GPDs have simple $\xi$-dependence (polynomials in $\xi$)

$\leftarrow$ should be possible to extrapolate!

Even moments of $H(x, \xi, t)$:

$$H_n(\xi, t) \equiv \int_{-1}^{1} dx x^{n-1} H(x, \xi, t) = \sum_{i=0}^{[\frac{n-1}{2}]} A_{n,2i}(t)\xi^{2i} + C_n(t)$$

$$= A_{n,0}(t) + A_{n,2}(t)\xi^2 + \ldots + A_{n,n-2}(t)\xi^{n-2} + C_n(t)\xi^n,$$
i.e. for example

\[ \int_{-1}^{1} dxxH(x, \xi, t) = A_{2,0}(t) + C_{2}(t)\xi^2. \]

- For \( n^{th} \) moment, need \( \frac{n}{2} + 1 \) measurements of \( H_n(\xi, t) \) for same \( t \) but different \( \xi \) to determine \( A_{n,2i}(t) \).
- GPDs @ \( \xi = 0 \) obtained from \( H_n(\xi = 0, t) = A_{n,0}(t) \)
- similar procedure exists for moments of \( \tilde{H} \)
QCD evolution

So far ignored! Can be easily included because

- For $t \ll Q^2$, leading order evolution $t$-independent

- For $\xi = 0$ evolution kernel for GPDs same as DGLAP evolution kernel

likewise:

- impact parameter dependent PDFs evolve such that different $b_\perp$ do not mix (as long as $\perp$ spatial resolution much smaller than $Q^2$)
above results consistent with QCD evolution:

\[
H(x, 0, -\Delta^2_{\perp}, Q^2) = \int d^2 b_{\perp} q(x, b_{\perp}, Q^2) e^{-i b_{\perp} \Delta_{\perp}} \\
\tilde{H}(x, 0, -\Delta^2_{\perp}, Q^2) = \int d^2 b_{\perp} \Delta q(x, b_{\perp}, Q^2) e^{-i b_{\perp} \Delta_{\perp}}
\]

where QCD evolution of \( H, \tilde{H}, q, \Delta q \) is described by DGLAP and is independent on both \( b_{\perp} \) and \( \Delta^2_{\perp} \), provided one does not look at scales in \( b_{\perp} \) that are smaller than \( 1/Q \).
suppression of crossed diagrams

Flow of the large momentum $q$ through typical diagrams contributing to the forward Compton amplitude. a) ‘handbag’ diagrams; b) ‘cat’s ears’ diagram. Diagram b) is suppressed at large $q$ due to the presence of additional propagators.
Form factor vs. charge distribution (non-rel.)

- define state that is localized in position space (center of mass frame)

\[ \left| \vec{R} = \vec{0} \right\rangle \equiv \mathcal{N} \int d^3 \vec{p} |\vec{p}\rangle \]

- define charge distribution (for this localized state)

\[ \rho(\vec{r}) \equiv \left\langle \vec{R} = \vec{0} \right| j^0(\vec{r}) \left| \vec{R} = \vec{0} \right\rangle \]
use translational invariance to relate to same matrix element that appears in def. of form factor

$$
\rho(\vec{r}) \equiv \left\langle \vec{R} = \vec{0} \right| j^0(\vec{r}) \left| \vec{R} = \vec{0} \right\rangle 
$$

$$
= |\mathcal{N}|^2 \int d^3 \vec{p} \int d^3 \vec{p}' \langle \vec{p}' | j^0(\vec{r}) | \vec{p} \rangle 
$$

$$
= |\mathcal{N}|^2 \int d^3 \vec{p} \int d^3 \vec{p}' \langle \vec{p}' | j^0(\vec{0}) | \vec{p} \rangle e^{i\vec{r} \cdot (\vec{p}-\vec{p}')}, 
$$

$$
= |\mathcal{N}|^2 \int d^3 \vec{p}' \int d^3 \vec{p} F \left( - (\vec{p}'_\perp - \vec{p}_\perp)^2 \right) e^{i\vec{b} \cdot (\vec{p}_\perp - \vec{p}'_\perp)} 
$$

$$
\rho(\vec{r}) = \int \frac{d^3 \Delta}{(2\pi)^3} F(-\Delta^2) e^{-i\vec{r}_\perp \cdot \Delta_\perp} 
$$
density interpretation of $q(x, b_\perp)$

- express quark-bilinear in twist-2 GPD in terms of light-cone ‘good’ component $\psi(+) \equiv \frac{1}{2} \gamma^- \gamma^+ \psi$

$$\bar{\psi}' \gamma^+ \psi = \bar{\psi}'(+) \gamma^+ \psi(+) = \sqrt{2} \psi'(+) \psi(+) .$$

- expand $\psi(+) \in$ terms of canonical raising and lowering operators

$$\psi(+) (x^-, x_\perp) = \int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \int \frac{d^2 k_\perp}{2\pi} \sum_s$$

$$\times [u(+) (k, s) b_s (k^+, k_\perp) e^{-ikx} + v(+) (k, s) d_s^\dagger (k^+, k_\perp) e^{ikx}]$$
with usual (canonical) equal light-cone time $x^+$ anti-commutation relations, e.g.

$$\{ b_r(k^+, k_{\perp}), b_s^\dagger(q^+, q_{\perp}) \} = \delta(k^+ - q^+) \delta(k_{\perp} - q_{\perp}) \delta_{rs}$$

and the normalization of the spinors is such that

$$\bar{u}_{(+)}(p, r) \gamma^+ u_{(+)}(p, s) = 2p^+ \delta_{rs}.$$

Note: $\bar{u}_{(+)}(p', r) \gamma^+ u_{(+)}(p, s) = 2p^+ \delta_{rs}$ for $p^+ = p'^+$, one finds for $x > 0$

$$q(x, b_{\perp}) = N' \sum_s \int \frac{d^2 k_{\perp}}{2\pi} \int \frac{d^2 k'_{\perp}}{2\pi} \langle p^+, 0_{\perp} | b_s^\dagger(xp^+, k'_{\perp})b_s(xp^+, k_{\perp}) | p^+, 0_{\perp} \rangle$$

$$\times e^{i b_{\perp} \cdot (k_{\perp} - k'_{\perp})}.$$
Switch to mixed representation: 

- momentum in longitudinal direction
- position in transverse direction

\[
\tilde{b}_s(k^+, x_\perp) \equiv \int \frac{d^2 k_\perp}{2\pi} b_s(k^+, k_\perp) e^{i k_\perp \cdot x_\perp}
\]

\[
q(x, b_\perp) = \sum_s \langle p^+, 0_\perp | \tilde{b}_s^\dagger(xp^+, b_\perp) \tilde{b}_s(xp^+, b_\perp) | p^+, 0_\perp \rangle^2
\]

\[
\geq 0.
\]
Boosts in nonrelativistic QM

\[ \vec{x}' = \vec{x} + \vec{v}t \quad t' = t \]

purely kinematical (quantization surface \( t = 0 \) inv.)

\( \leftarrow \) 1. boosting wavefunctions very simple

\[ \Psi_{\vec{v}}(\vec{p}_1, \vec{p}_2) = \Psi_0(\vec{p}_1 - m_1 \vec{v}, \vec{p}_2 - m_2 \vec{v}). \]

2. dynamics of center of mass

\[ \vec{R} \equiv \sum_i x_i \vec{r}_i \quad \text{with} \quad x_i \equiv \frac{m_i}{M} \]

decouples from the internal dynamics

Physical Interpretation for the Generalized Parton Distributions \( H(x, 0, -\Delta^2_{\perp}) \) and \( E(x, 0, -\Delta^2_{\perp}) \) – p.43/55
Relativistic Boosts

\[ t' = \gamma \left( t + \frac{v}{c^2} z \right), \quad z' = \gamma \left( z + vt \right), \quad x'_\perp = x_\perp \]

generators satisfy Poincaré algebra:

\[
\begin{align*}
[J^{\mu}, P^\nu] &= 0 \\
[J^{\mu\nu}, P^\rho] &= i \left( g^{\nu\rho} P^\mu - g^{\mu\rho} P^\nu \right) \\
[J^{\mu\nu}, J^{\rho\lambda}] &= i \left( g^{\mu\lambda} J^{\nu\rho} + g^{\nu\rho} J^{\mu\lambda} - g^{\mu\rho} J^{\nu\lambda} - g^{\nu\lambda} J^{\mu\rho} \right)
\end{align*}
\]

rotations: \( M_{i,j} = \varepsilon_{ijk} J_k \),

boosts: \( M_{i0} = K_i \).
Galilean subgroup of $\perp$ boosts

introduce generator of $\perp$ ‘boosts’:

$$B_x \equiv M^{+x} = \frac{K_x + J_y}{\sqrt{2}} \quad B_y \equiv M^{+y} = \frac{K_y - J_x}{\sqrt{2}}$$

Poincaré algebra $\implies$ commutation relations:

$$[J_3, B_k] = i \varepsilon_{kl} B_l \quad [P_k, B_l] = -i \delta_{kl} P^+$$

$$[P^-, B_k] = -i P_k \quad [P^+, B_k] = 0$$

with $k, l \in \{x, y\}$, $\varepsilon_{xy} = -\varepsilon_{yx} = 1$, and $\varepsilon_{xx} = \varepsilon_{yy} = 0$. 
Together with \([J_z, P_k] = i\varepsilon_{kl} P_l\), as well as

\[
\begin{align*}
[P^-, P_k] &= [P^-, P^+] = [P^-, J_z] = 0 \\
[P^+, P_k] &= [P^+, B_k] = [P^+, J_z] = 0.
\end{align*}
\]

Same as commutation relations among generators of nonrel. boosts, translations, and rotations in x-y plane, provided one identifies

- \(P^- \rightarrow \text{Hamiltonian}\)
- \(P_\perp \rightarrow \text{momentum in the plane}\)
- \(P^+ \rightarrow \text{mass}\)
- \(L_z \rightarrow \text{rotations around } z\text{-axis}\)
- \(B_\perp \rightarrow \text{generator of boosts in the plane}\)

back to discussion
Consequences

- many results from NRQM carry over to \( \perp \) boosts in IMF, e.g.

- \( \perp \) boosts kinematical

\[
\psi_{\Delta_\perp}(x, k_\perp) = \psi_{0_\perp}(x, k_\perp - x\Delta_\perp) \\
\psi_{\Delta_\perp}(x, k_\perp, y, l_\perp) = \psi_{0_\perp}(x, k_\perp - x\Delta_\perp, y, l_\perp - y\Delta_\perp)
\]

- Transverse center of momentum \( \mathbf{R}_\perp \equiv \sum_i x_i \mathbf{r}_\perp,i \) plays role similar to NR center of mass, e.g.

\[
|\psi_{loc}\rangle \equiv \int d^2 p_\perp |p^+, \mathbf{p}_\perp\rangle \text{ corresponds to state with } \mathbf{R}_\perp = 0_\perp.
\]
Center of Momentum

- field theoretic definition

\[ p^+ R_\perp \equiv \int dx^- \int d^2 x_\perp T^{++}(x) x_\perp = M^{+\perp} \]

- \( M^{+\perp} = B^{\perp} \) generator of transverse boosts
- parton representation:

\[ R_\perp = \sum_i x_i r_{\perp,i} \]

\( (x_i = \text{momentum fraction carried by } i^{th} \text{ parton}) \)

back
Poincaré algebra:

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\left[ P^\mu, P^\nu \right] &= 0 \\
\left[ M^{\mu\nu}, P^\rho \right] &= i \left( g^{\nu\rho} P^\mu - g^{\mu\rho} P^\nu \right) \\
\left[ M^{\mu\nu}, M^{\rho\lambda} \right] &= i \left( g^{\mu\lambda} M^{\nu\rho} + g^{\nu\rho} M^{\mu\lambda} - g^{\mu\rho} M^{\nu\lambda} - g^{\nu\lambda} M^{\mu\rho} \right)
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rotations: \( M_{ij} = \varepsilon_{ijk} J_k \),   \[ \text{boosts: } M_{i0} = K_i. \]
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with \( k, l \in \{x, y\} \), \( \varepsilon_{xy} = -\varepsilon_{yx} = 1 \), and \( \varepsilon_{xx} = \varepsilon_{yy} = 0 \).
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$$
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\[
|p^+, R_\perp = 0_\perp\rangle \equiv \int d^2 p_\perp |p^+, p_\perp\rangle \text{ corresponds to state with } R_\perp = 0_\perp.
\]

back
Proof that $B_{\perp} |p^+, R_{\perp} = 0_{\perp} \rangle = 0$

- Use

$$e^{-i v_{\perp} \cdot B_{\perp}} |p^+, p_{\perp}, \lambda \rangle = |p^+, p_{\perp} + p^+ v_{\perp}, \lambda \rangle$$

$$\Rightarrow$$

$$e^{-i v_{\perp} \cdot B_{\perp}} \int d^2 p_{\perp} |p^+, p_{\perp}, \lambda \rangle = \int d^2 p_{\perp} |p^+, p_{\perp}, \lambda \rangle$$

$$\Rightarrow$$

$$B_{\perp} \int d^2 p_{\perp} |p^+, p_{\perp}, \lambda \rangle = 0$$
Example

- Ansatz: \( H_q(x, 0, -\Delta_\perp^2) = q(x)e^{-a\Delta_\perp^2(1-x)\ln \frac{1}{x}}. \)

\[
q(x, b_\perp^2) = q(x) \frac{1}{4\pi a(1-x)\ln \frac{1}{x}} e^{-\frac{b_\perp^2}{4a(1-x)\ln \frac{1}{x}}}
\]
$q(x, b_\perp)$