Topological Quantum Information, Khovanov Homology and the Jones Polynomial

Louis H. Kauffman
Department of Mathematics, Statistics
and Computer Science (m/c 249)
851 South Morgan Street
University of Illinois at Chicago
Chicago, Illinois 60607-7045
<kauffman@uic.edu>

Arxiv: 1001.0354

Background Ideas
and
References

Topological Quantum Information Theory
(with Sam Lomonaco)

quant-ph/0603131 and quant-ph/0606114

Spin Networks and Anyonic TQC

arXiv:0805.0339 **Quantum Knots and Mosaics**

arXiv:0910.5891

arXiv:0804.4304 The Fibonacci Model and the Temperley-Lieb Algebra

arXiv:0706.0020 A 3-Stranded Quantum Algorithm for the Jones Polynomial

arXiv:0909.1080 NMR Quantum Calculations of the Jones Polynomial
Authors: Raimund Marx, Amr Fahmy, Louis Kauffman,

Samuel Lomonaco, Andreas Spörl, Nikolas Pomplun, John Myers, Steffen J. Glaser

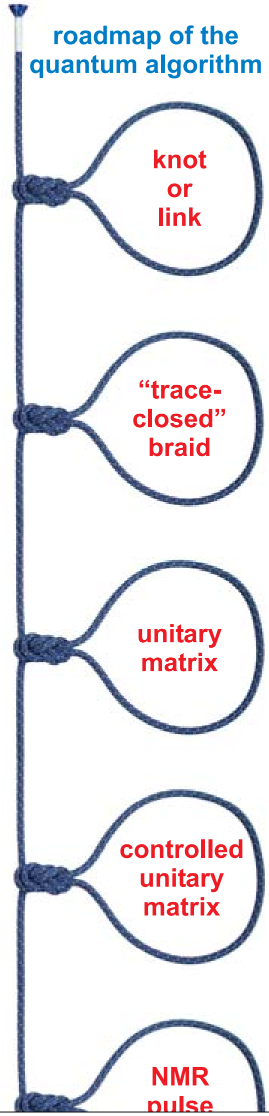
arXiv: 0909.1672 Anyonic topological quantum computation and the virtual braid group. H. Dye and LK.

Untying Knots by NMR: first experimental implementation of a quantum algorithm for approximating the Jones polynomial

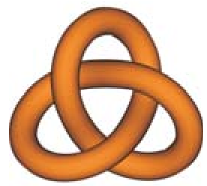
Raimund Marx[1], Andreas Spörl[1], Amr F. Fahmy[2], John M. Myers[3], Louis H. Kauffman[4], Samuel J. Lomonaco, Jr.[5], Thomas-Schulte-Herbrüggen[1], and Steffen J. Glaser[1]

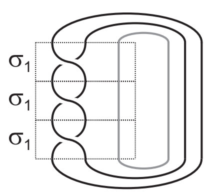
[1]Department of Chemistry, Technical University Munich, Lichtenbergstr. 4, 85747 Garching, Germany
[2]Harvard Medical School, 25 Shattuck Street, Boston, MA 02115, U.S.A.
[3]Gordon McKay Laboratory, Harvard University, 29 Oxford Street, Cambridge, MA 02138, U.S.A.
[4]University of Illinois at Chicago, 851 S. Morgan Street, Chicago, IL 60607-7045, U.S.A.
[5]University of Maryland Baltimore County, 1000 Hilltop Circle, Baltimore, MD 21250, U.S.A.

roadmap of the quantum algorithm

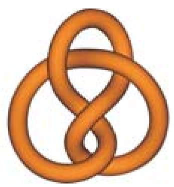


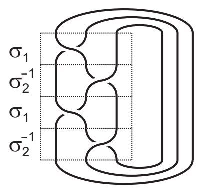
example #1 Trefoil



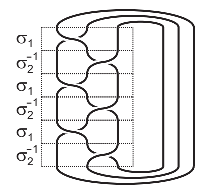


example #2 Figure-Eight





example #3 Borromean rings



$$U_{Trefoil} = (U_1)^3 \qquad U_{Figure-Eight} = (U_2^{-1}\cdot U_1)^2 \qquad U_{Borrom.R.} = (U_2^{-1}\cdot U_1)^3$$

$$U_1 = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & -e^{i\theta}\dfrac{\sin(4\theta)}{\sin(2\theta)} + e^{-i\theta} \end{pmatrix}$$

$$U_2 = \begin{pmatrix} -e^{i\theta}\dfrac{\sin(6\theta)}{\sin(4\theta)} + e^{-i\theta} & -e^{i\theta}\dfrac{\sqrt{\sin(6\theta)\sin(2\theta)}}{\sin(4\theta)} \\ -e^{i\theta}\dfrac{\sqrt{\sin(6\theta)\sin(2\theta)}}{\sin(4\theta)} & -e^{i\theta}\dfrac{\sin(2\theta)}{\sin(4\theta)} + e^{-i\theta} \end{pmatrix}$$

Step #1:
from the 2x2 matrix U
to the 4x4 matrix cU:

$$cU = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}$$

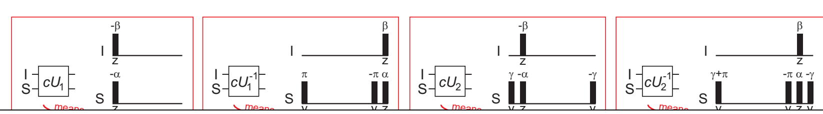
Step #2:
application of cU on the
NMR product operator I_{1x} :

$$cU\, I_{1x}\, cU^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}^{\dagger} = \frac{1}{2}\begin{pmatrix} 0 & U^{\dagger} \\ U & 0 \end{pmatrix}$$

Step #3:
measurement
of I_{1x} and I_{1y} :

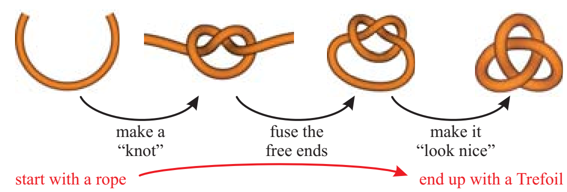
$$tr\left\{ I_{1x}\,\frac{1}{2}\begin{pmatrix} 0 & U^{\dagger} \\ U & 0 \end{pmatrix}\right\} = \frac{1}{2}\Re(tr\{U\})$$

$$tr\left\{ I_{1y}\,\frac{1}{2}\begin{pmatrix} 0 & U^{\dagger} \\ U & 0 \end{pmatrix}\right\} = \frac{1}{2}\Im(tr\{U\})$$



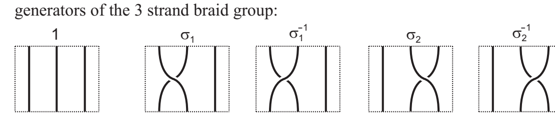
A knot is defined as a closed, non-self-intersecting curve that is embedded in three dimensions.

example: "construction" of the Trefoil knot:



J. W. Alexander proved, that any knot can be represented as a closed braid (polynomial time algorithm)

generators of the 3 strand braid group:



It is well known in knot theory, how to obtain the unitary matrix representation of all generators of a given braid goup (see "Temperley-Lieb algebra" and "path model representation"). The unitary matrices U_1 and U_2, corresponding to the generators σ_1 and σ_2 of the 3 strand braid group are shown on the left, where the variable "θ" is related to the variable "A" of the Jones polynomial by: $A = e^{-i\theta_{rad}}$. The unitary matrix representations of σ_1^{-1} and σ_2^{-1} are given by U_1^{-1} and U_2^{-1}.

The knot or link that was expressed as a product of braid group generators can therefore also be expressed as a product of the corresponding unitary matrices.

Instead of applying the unitary matrix U, we apply it's controlled variant cU. This matrix is especially suited for NMR quantum computers [4] and other thermal state expectation value quantum computers: you only have to apply cU to the NMR product operator I_{1x} and measure I_{1x} and I_{1y} in order to obtain the trace of the original matrix U.

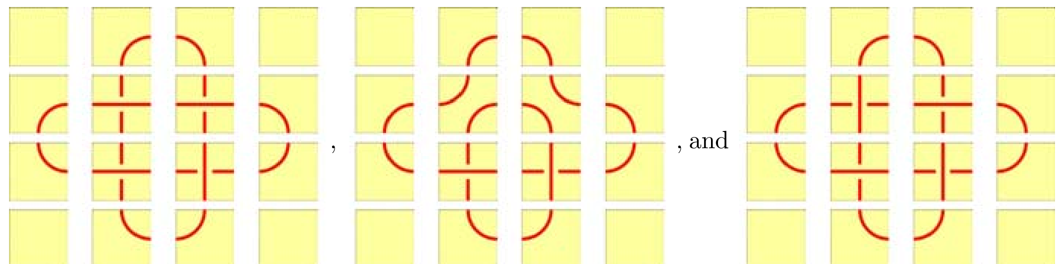
Independent of the dimension of matrix U you only need ONE extra qubit for the implementation of cU as compared to the implementation of U itself.

The measurement of I_{1x} and I_{1y} can be accomplished in one single-scan experiment.

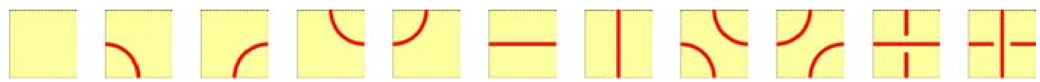
All knots and links can be expressed as a product of braid group generators (see above). Hence the corresponding NMR pulse sequence can also be expressed as a sequence of NMR pulse sequence blocks, where each block corresponds to the controlled unitary matrix cU of one braid group generator.

Quantum knots and mosaics

with
Sam
Lomonaco

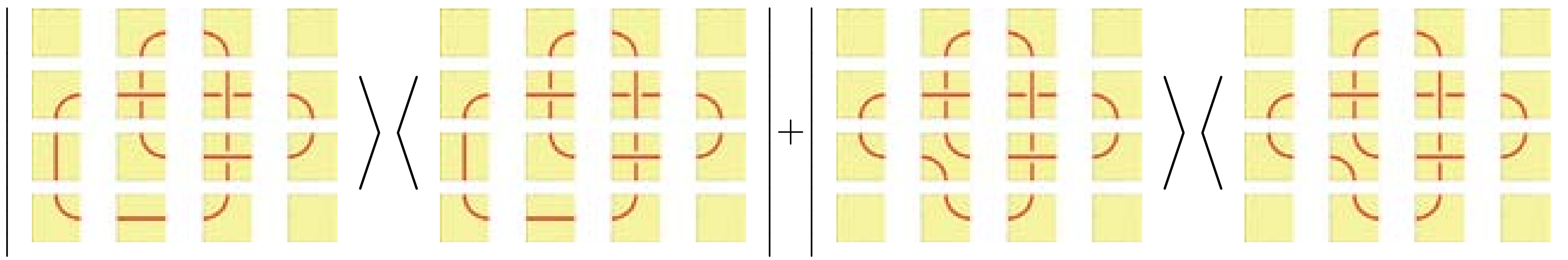


Each of these knot mosaics is a string made up of the following 11 symbols



called *mosaic tiles*.

Each mosaic is a tensor product of elementary tiles.

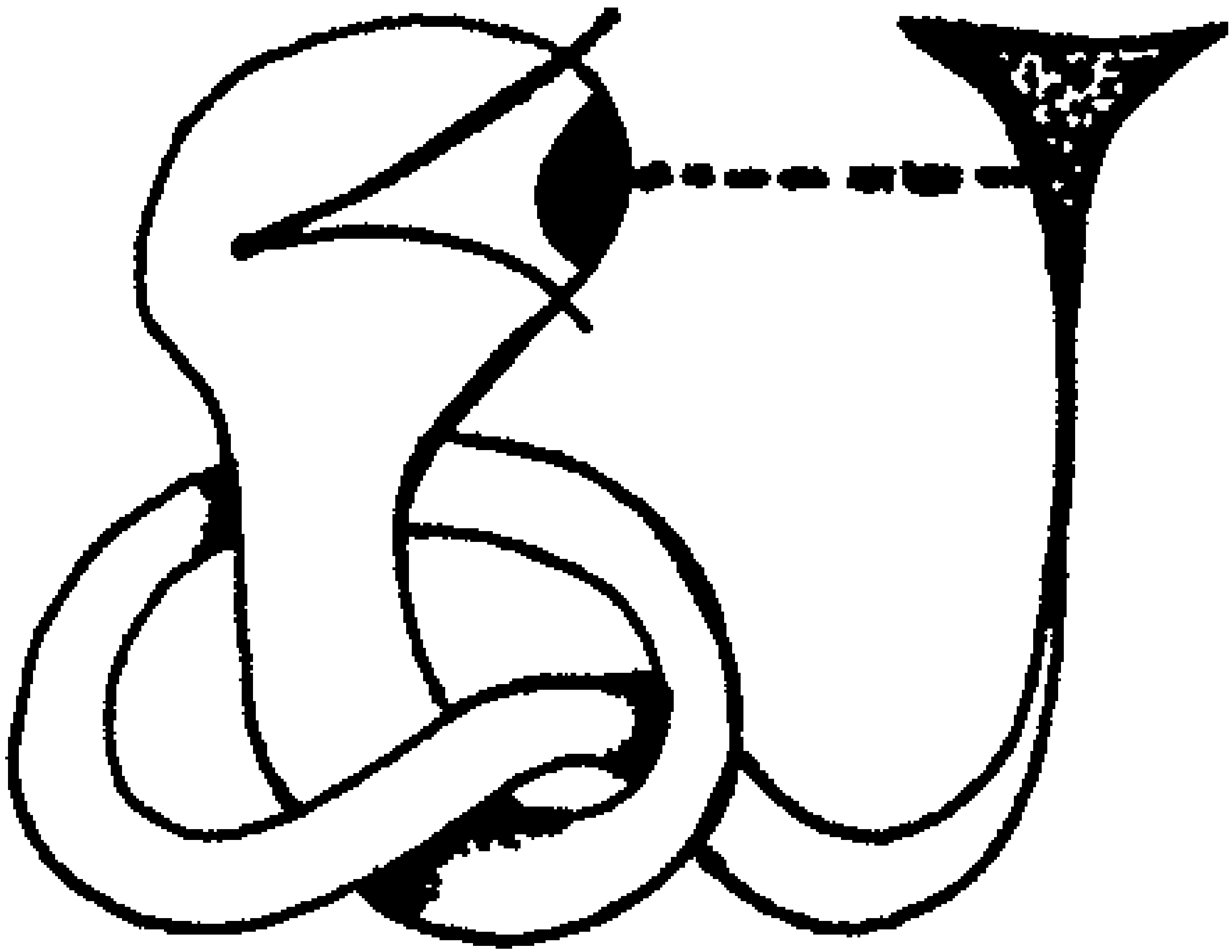
$\Omega =$ 

This observable is a quantum knot invariant for 4x4 tile space. Knots have characteristic invariants in nxn tile space.

Superpositions of combinatorial knot configurations are seen as quantum states.

The Grand Generalization

Universe as a Quantum Knot: Self-Excited Circuit Producing its Own Context



Papers on Quantum Computing, Knots and Khovanov Homology

arXiv:1001.0354

Title: Topological Quantum Information, Khovanov Homology and the Jones Polynomial
Authors: Louis H. Kauffman

arXiv:0907.3178

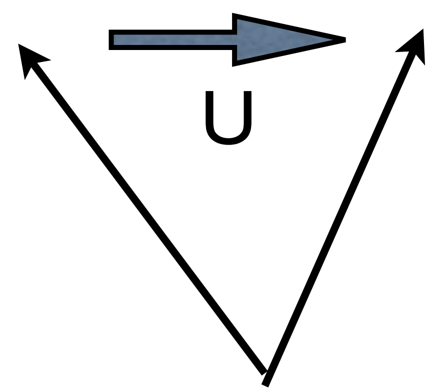
Title: Remarks on Khovanov Homology and the Potts Model
Authors: Louis H. Kauffman

The ideas here are related with structure of quantum knots.

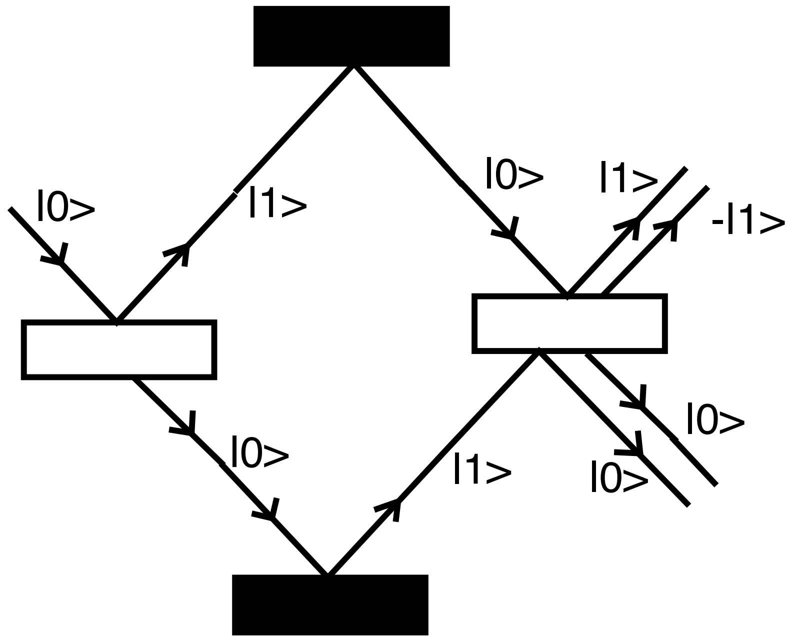
Quantum Mechanics in a Nutshell

0. A state of a physical system corresponds to a unit vector |S> in a complex vector space.

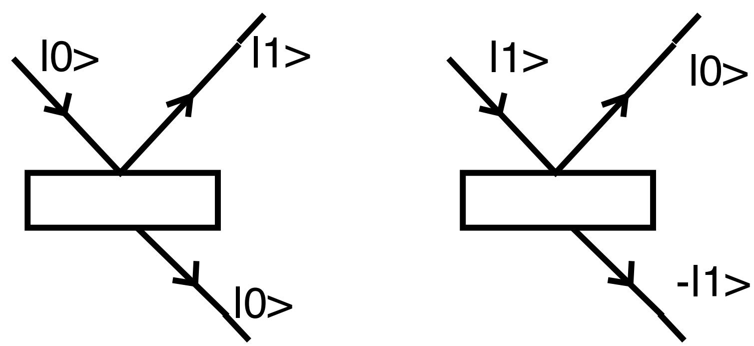
1. (measurement free) Physical processes are modeled by unitary transformations applied to the state vector: |S> -----> U|S>



2. If $|S\rangle = z_1|1\rangle + z_2|2\rangle + \ldots + z_n|n\rangle$ in a measurement basis {|1>,|2>,...,|n>}, then measurement of |S> yields |i> with probability |z|^2.



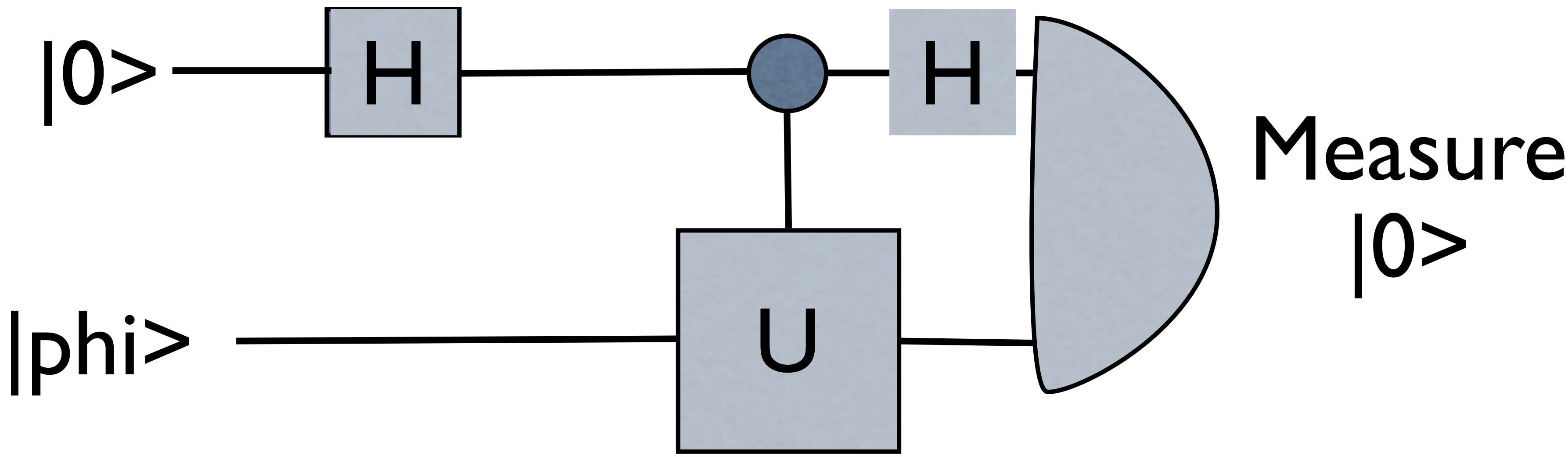
Mach-Zender Interferometer



$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} /\mathrm{Sqrt}(2) \qquad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

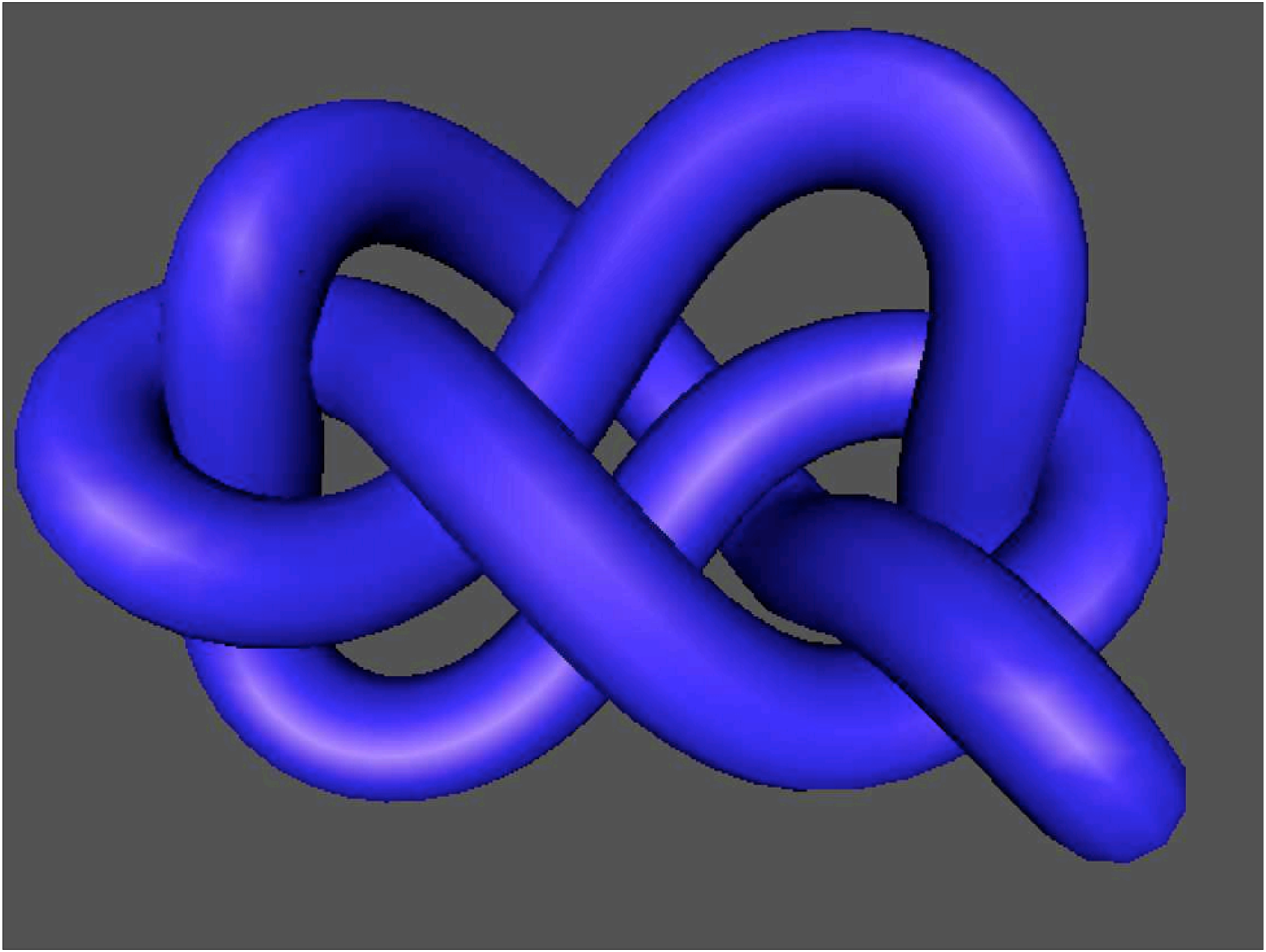
$$HMH = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

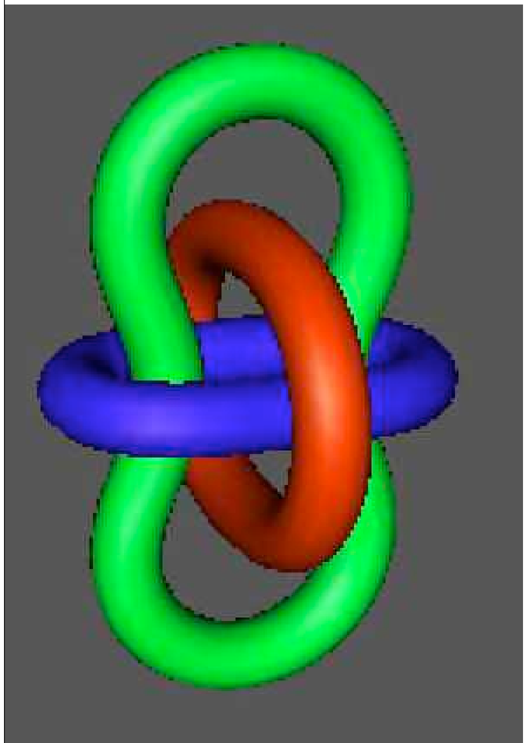
Hadamard Test



$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} / \mathrm{Sqrt}(2)$$

|0> occurs with probability
1/2 + Re[<phi|U|phi>]/2.





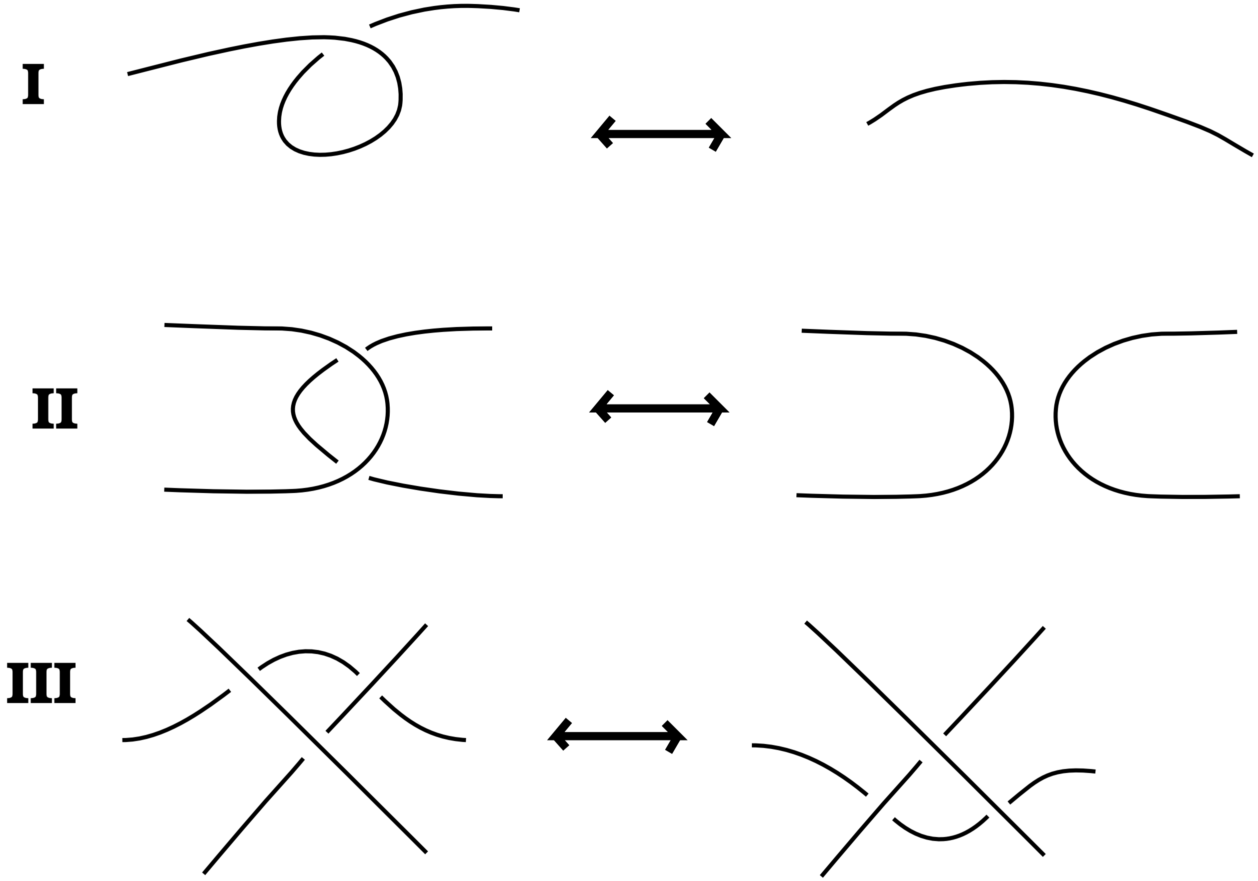
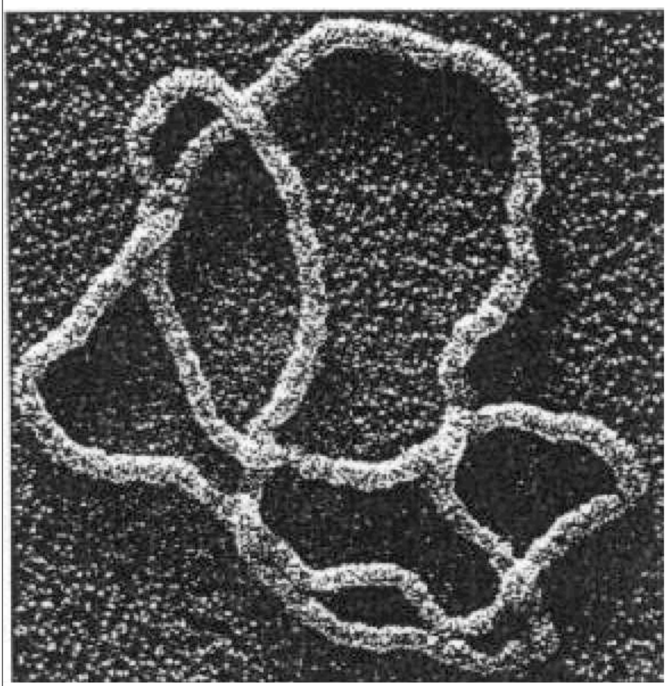
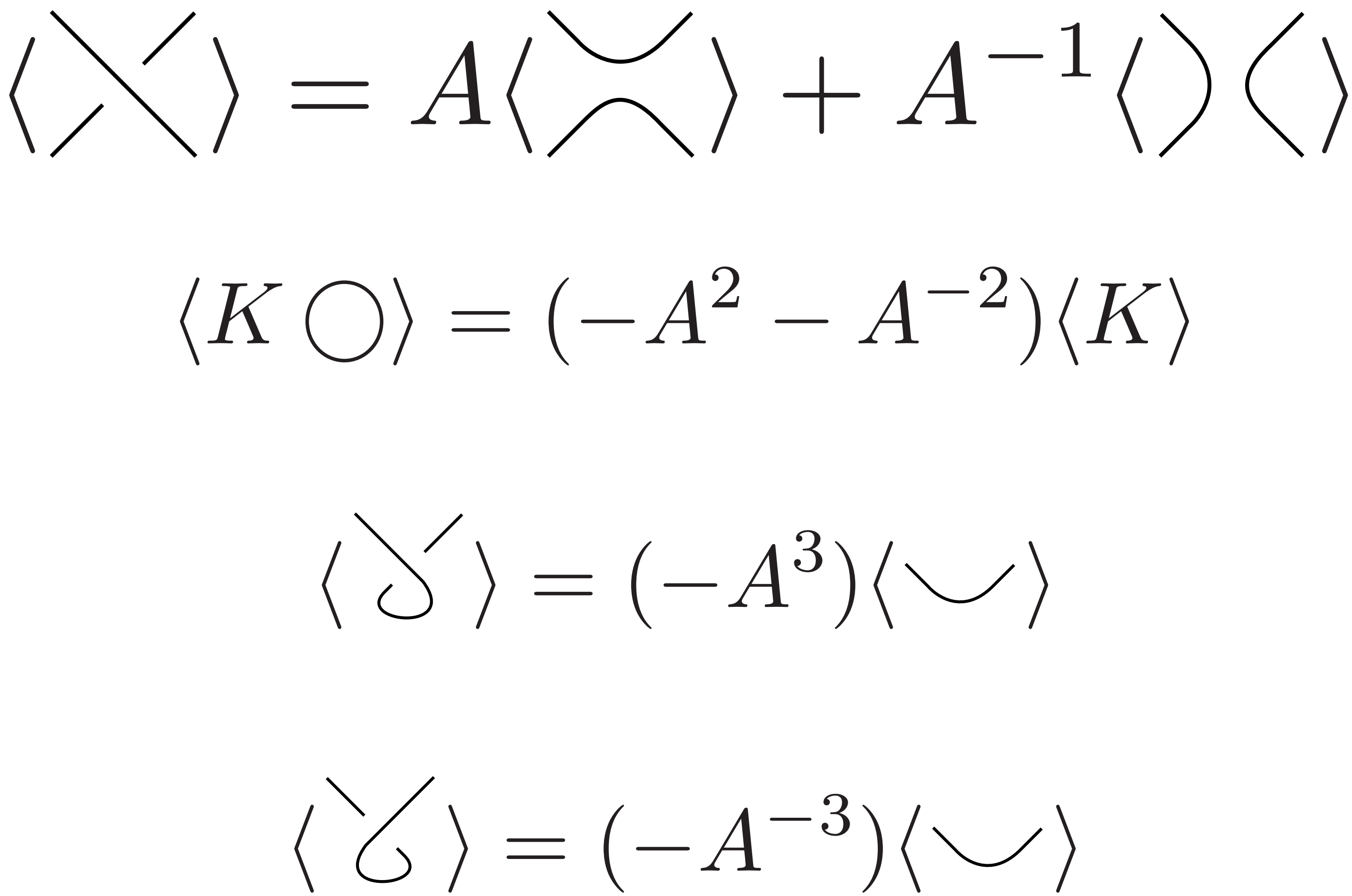


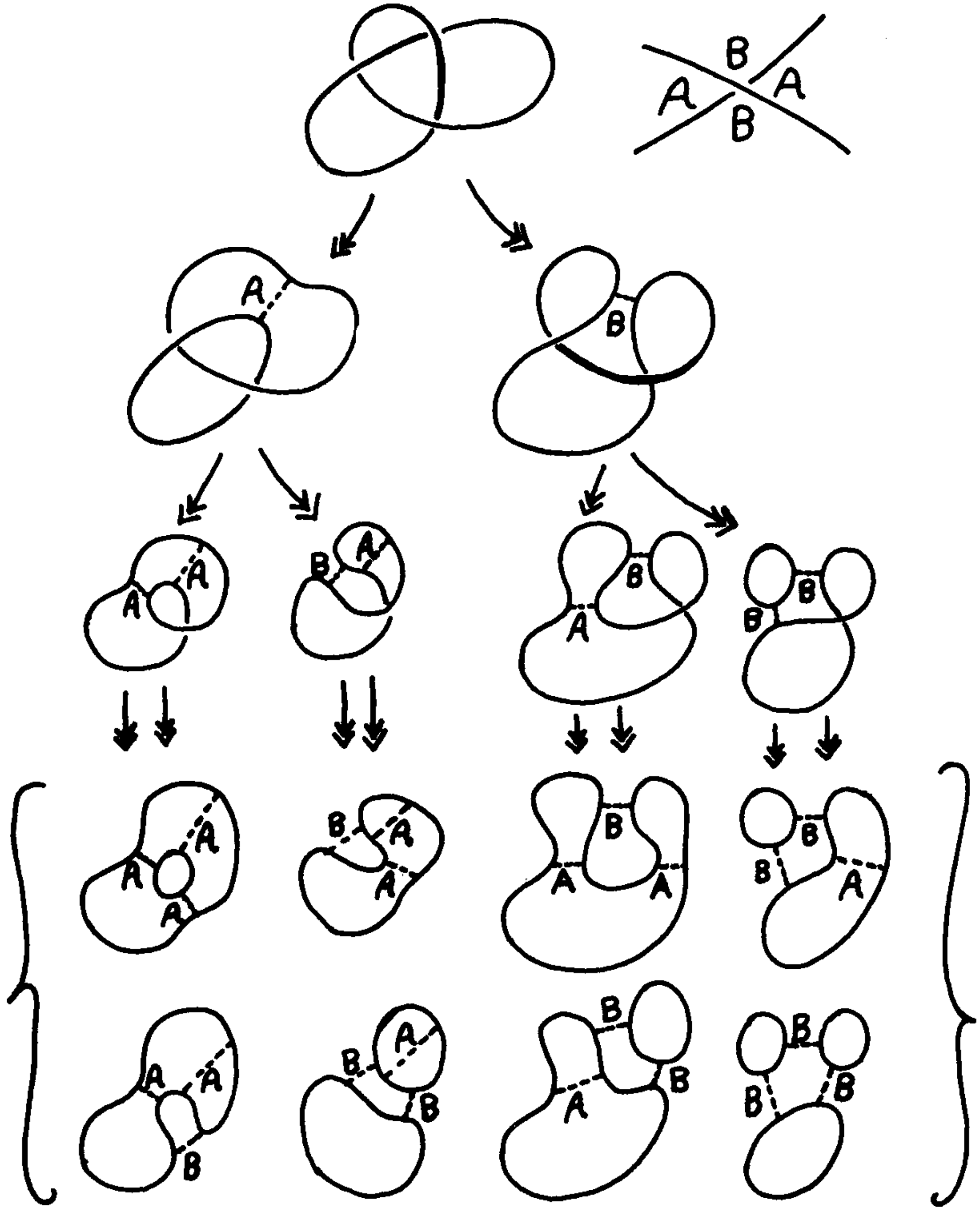
Figure 2 - The Reidemeister Moves.



Reidemeister Moves reformulate knot theory in terms of graph combinatorics.

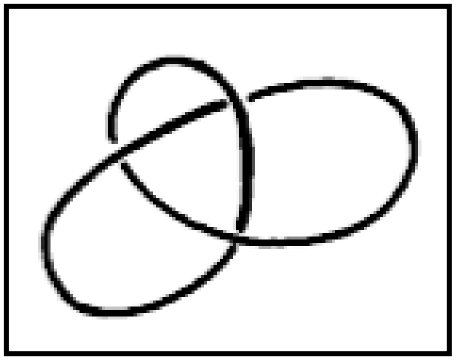
Bracket Polynomial Model for the Jones Polynomial

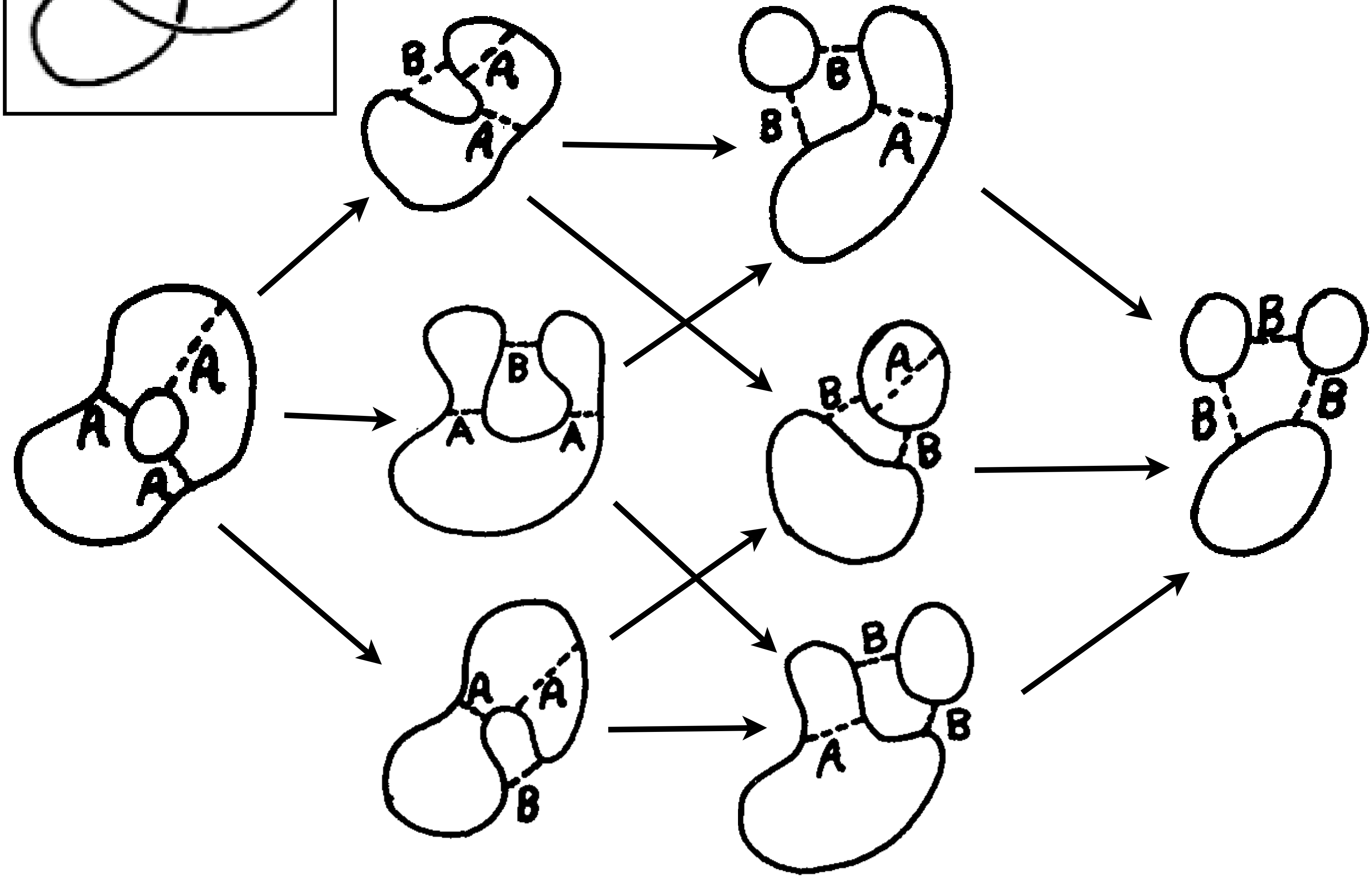




B
A
A
B
A
B
A
A
B
A
A
B
B
B
A
A
A
A
B
A
A
B
B
A
A
A
B
A
A
A
A
B
A
B
B
A
B
B
B
B
B
B

The Khovanov Complex





CATEGORIFICATION

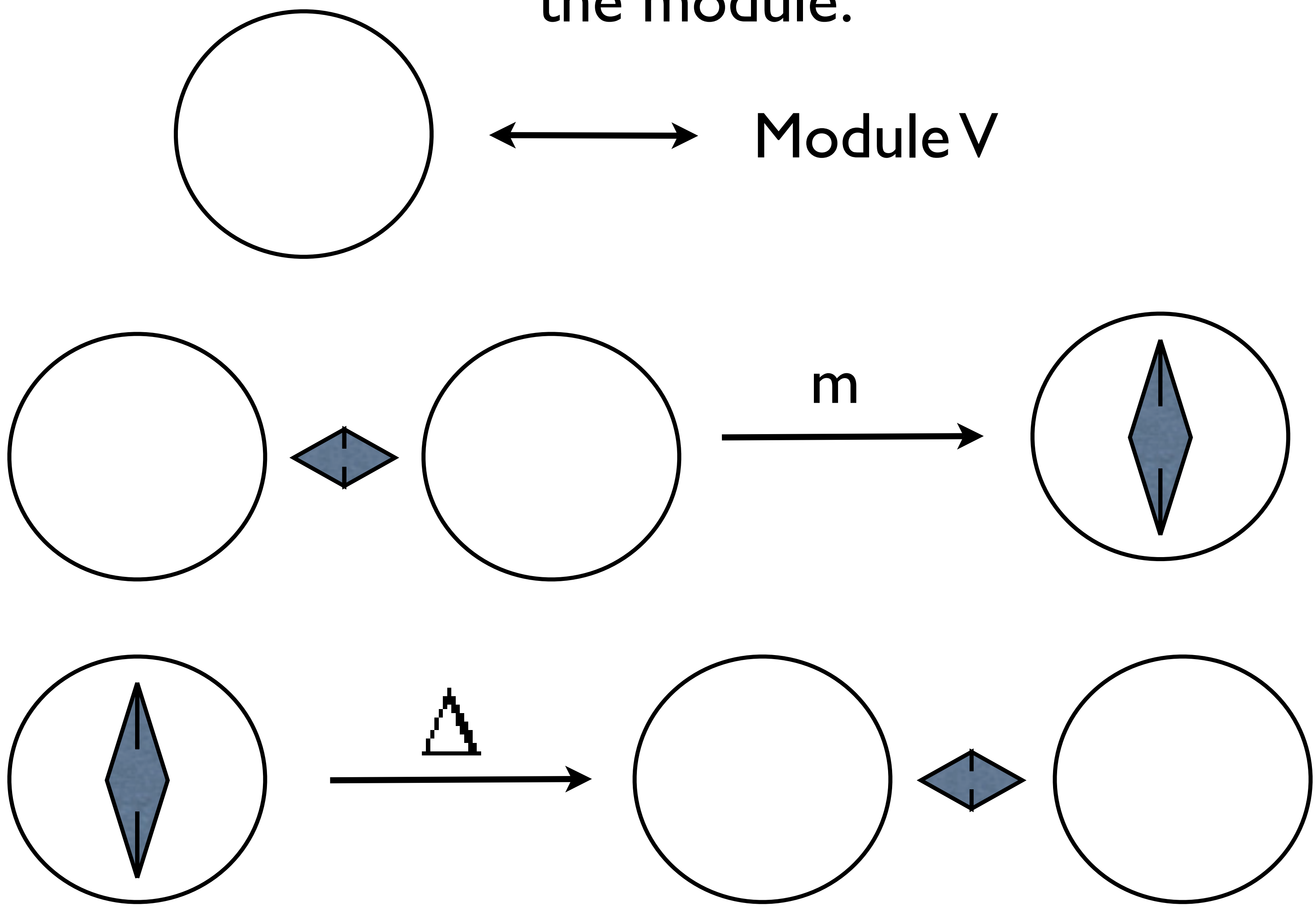
View the previous slide of states of the bracket expansion as a CATEGORY.

The cubical shape of this category suggests making a homology theory.

In order to make a non-trivial homology theory we need a functor from this category of states to a module category. Each state loop will map to a module V. Unions of loops will map to tenor products of this module.

We will describe how this comes about after looking at the bracket polynomial in more detail.

The Functor from the cubical category to the module category demands multiplication and comultiplication in the module.

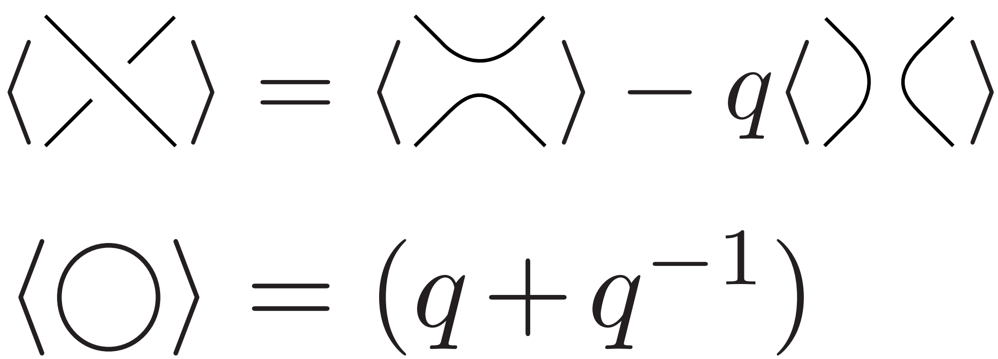


Reformulating the Bracket

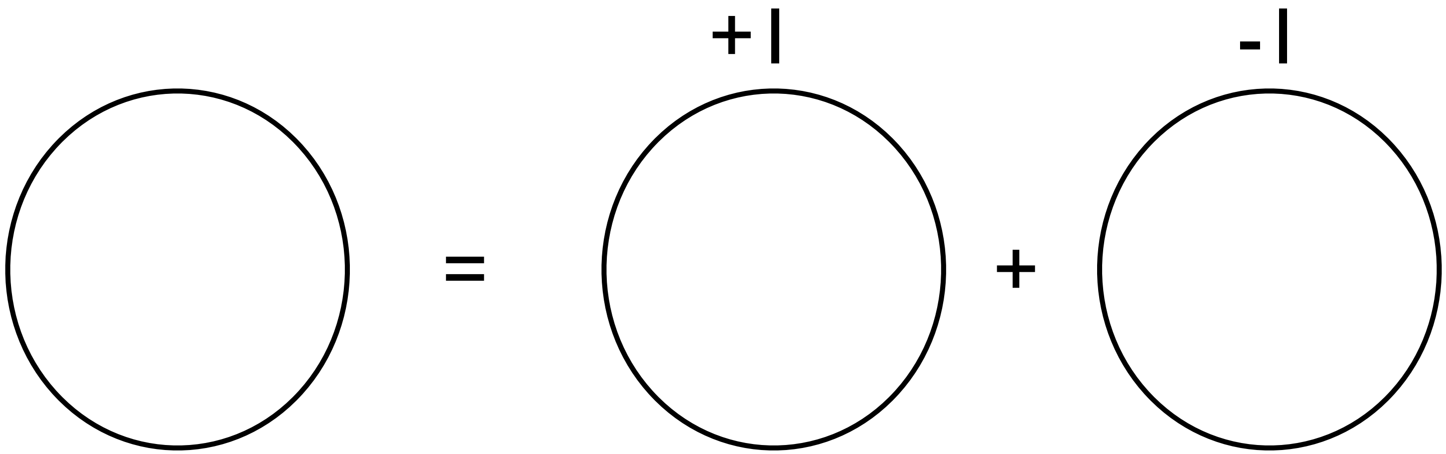
Let c(K) = number of crossings on link K.

Form $A^{-c(K)}$ <K> and replace A by $-q^{-2}$.

Then the skein relation for <K> will be replaced by:



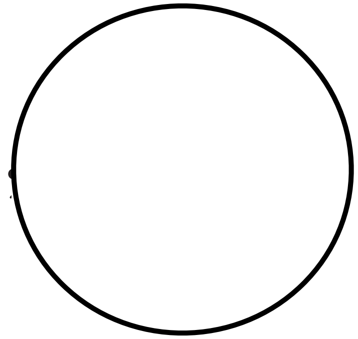
Use enhanced states by labeling each loop with
+1 or -1.



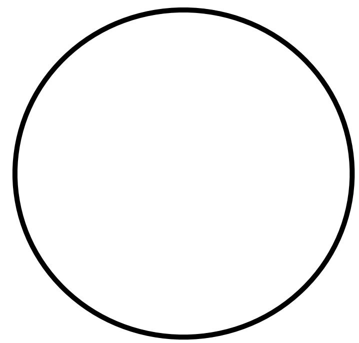
$$\longleftrightarrow \quad q + q^{-1}$$

Enhanced States

$$q^{-1} \Longleftrightarrow -1 \Longleftrightarrow X.$$

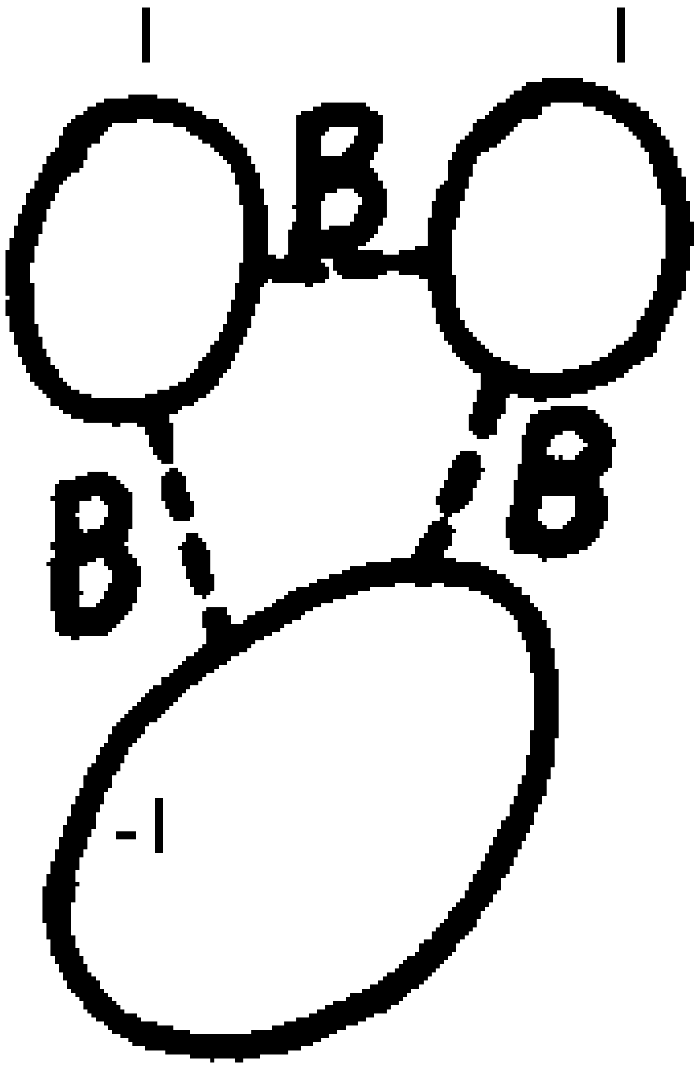


$$q^{+1} \Longleftrightarrow +1 \Longleftrightarrow 1$$



For reasons that will soon become apparent, we let -1 be denoted by X and +1 be denoted by 1.

(The module V will be generated by 1 and X.)



An enhanced state
that contributes

[(q)(q)(1/q)] [(-q) (-q) (-q)]

1 1 -1 **B B B**

to the revised
bracket state sum.

Enhanced State Sum Formula for the Bracket

$$\langle K \rangle = \sum_{s} q^{j(s)}(-1)^{i(s)}$$

A Quantum Statistical Model for the Bracket Polynonmial.

Let C(K) denote a Hilbert space with basis |s> where s runs over the enhanced states of a knot or link diagram K.

We define a unitary transformation.

$$U : \mathcal{C}(K) \longrightarrow \mathcal{C}(K)$$

$$U|s\rangle = (-1)^{i(s)} q^{j(s)} |s\rangle$$

q is chosen on the unit circle in the complex plane.

$$|\psi\rangle = \sum_s |s\rangle$$

Lemma. The evaluation of the bracket polynomial is given by the following formula

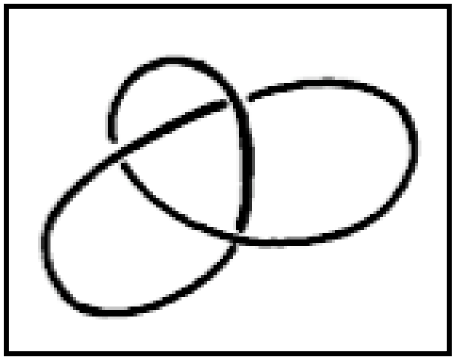
$$\langle K \rangle = \langle \psi | U | \psi \rangle .$$

This gives a new quantum algorithm for the Jones polynomial (via Hadamard Test).

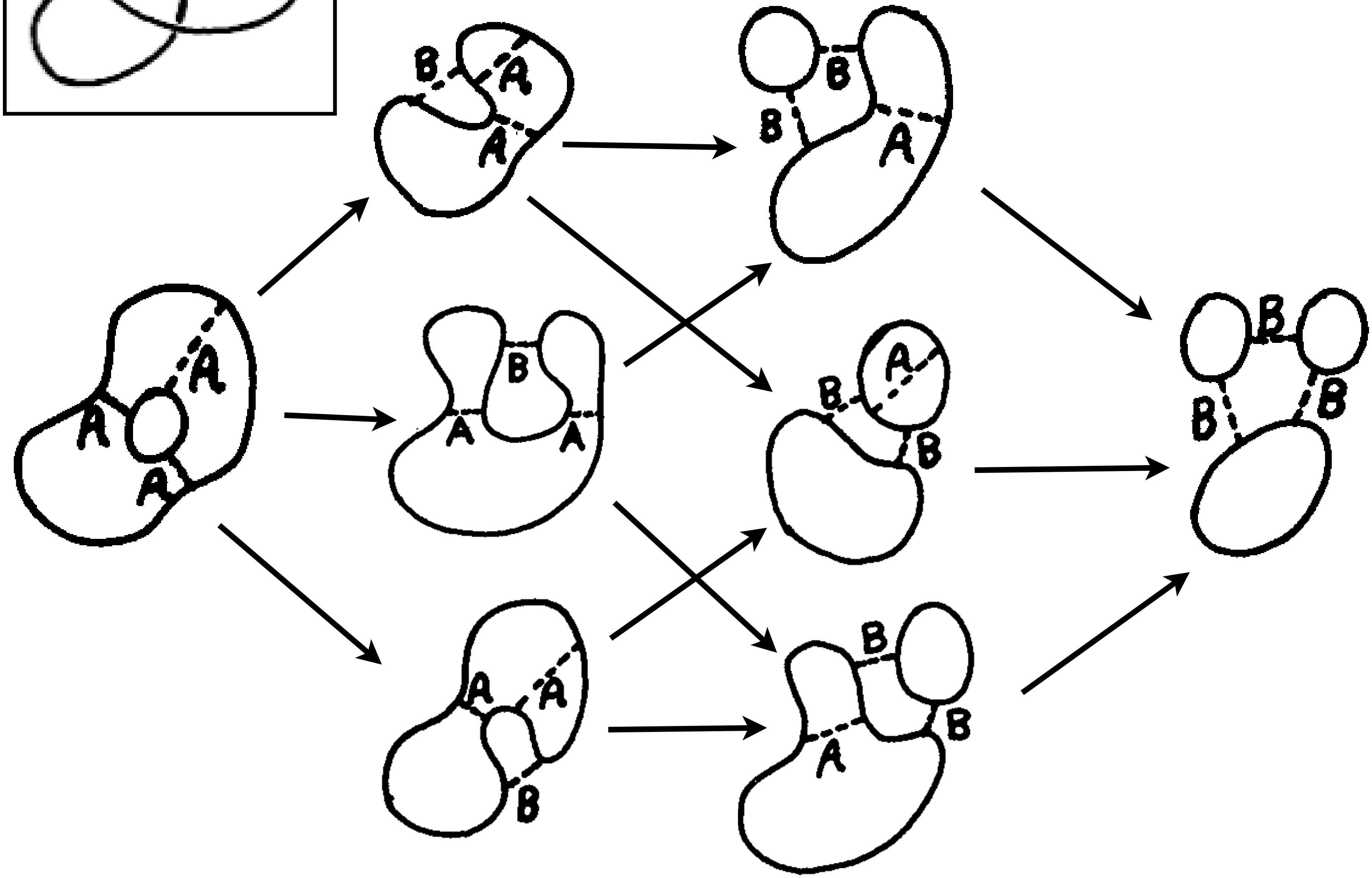
Khovanov Homology - Jones Polynomial as an Euler Characteristic

Two key motivating ideas are involved in finding the Khovanov invariant. First of all, one would like to *categorify* a link polynomial such as $\langle K \rangle$. There are many meanings to the term categorify, but here the quest is to find a way to express the link polynomial as a *graded Euler characteristic* $\langle K \rangle = \chi_q \langle H(K) \rangle$ for some homology theory associated with $\langle K \rangle$.

We will formulate Khovanov Homology
in the context of our quantum statistical model for the bracket polynomial.

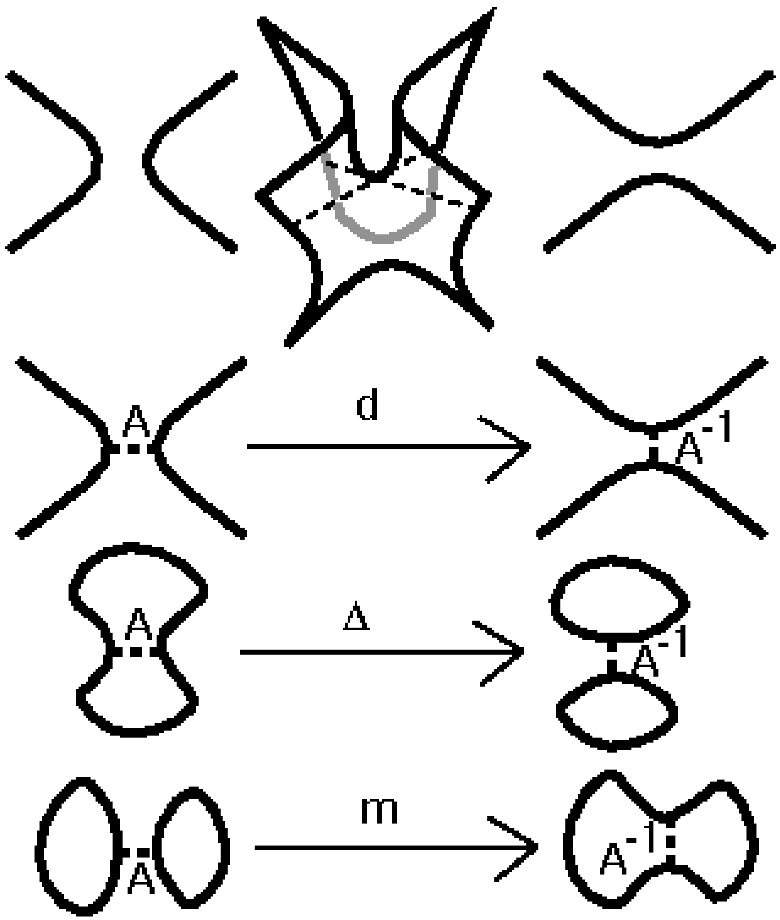


The Khovanov Complex



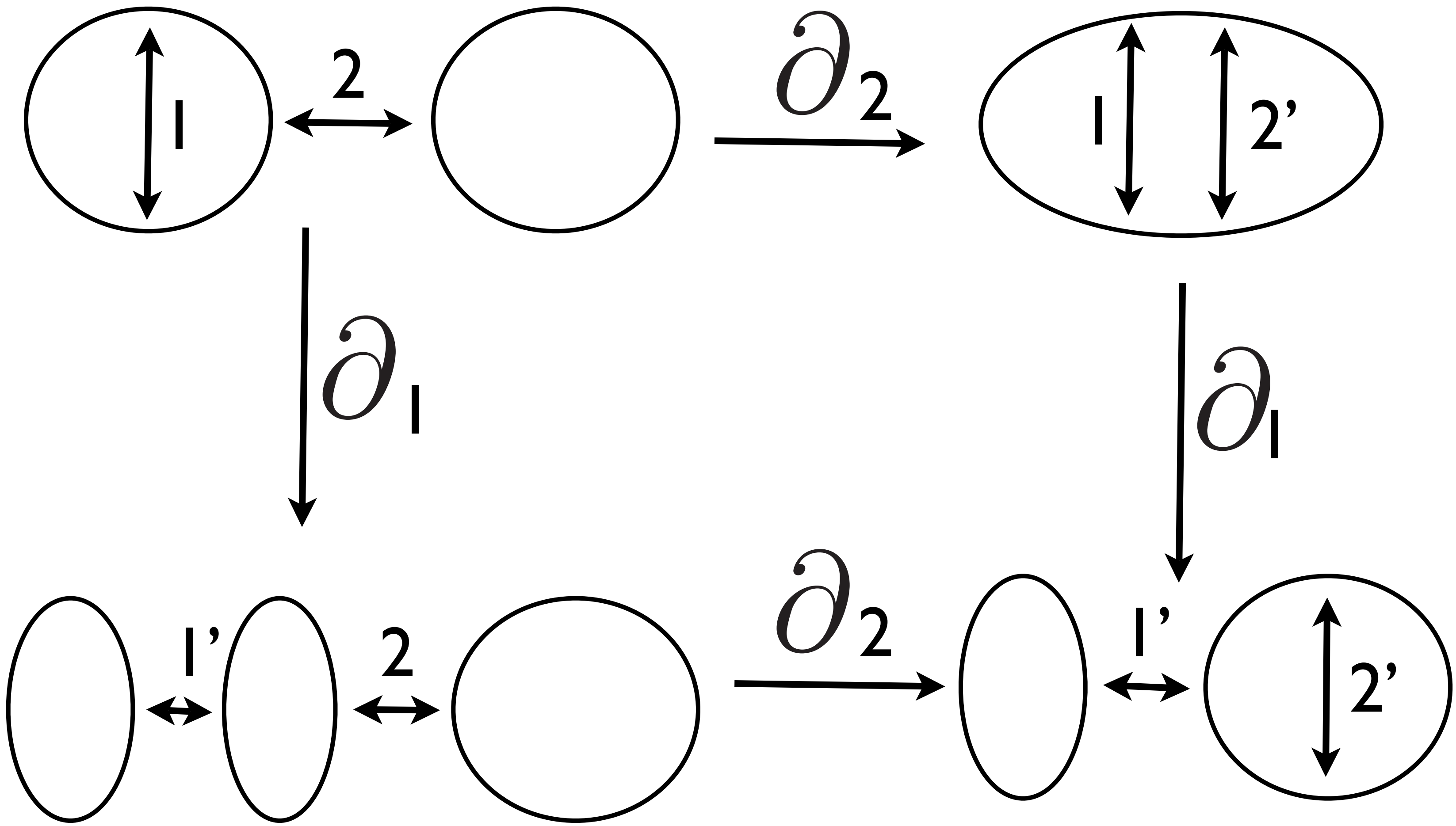
$$\partial(s) = \sum_{\tau} \partial_{\tau}(s)$$

The boundary is a sum of partial differentials corresponding to resmoothings on the states.



Each state loop is a module.

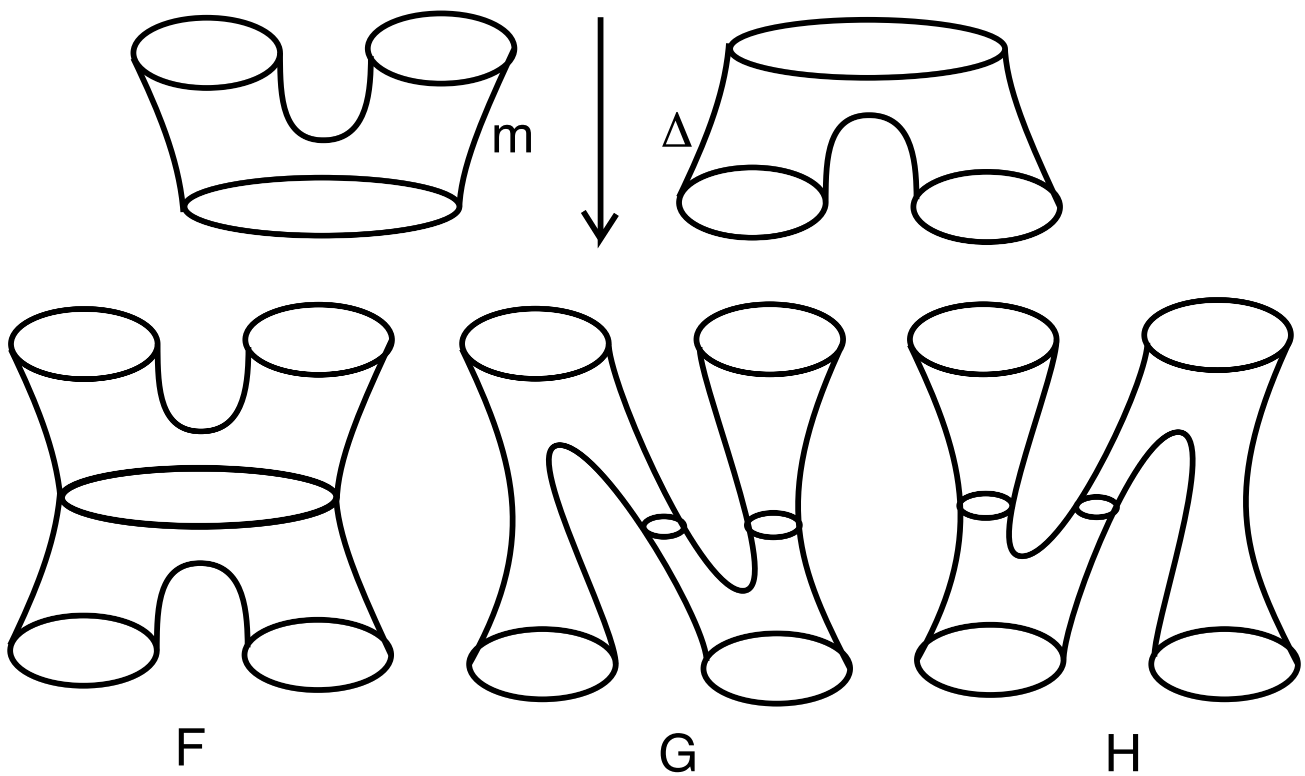
A collection of state loops corresponds to a tensor product of these modules.



For d^2 =0, want partial boundaries to commute.

The commutation of the partial boundaries leads to a structure of Frobenius algebra for the algebra associated to a state circle.

In our context this means that the qubit space V spanned by 1 and X is a Frobenius algebra.

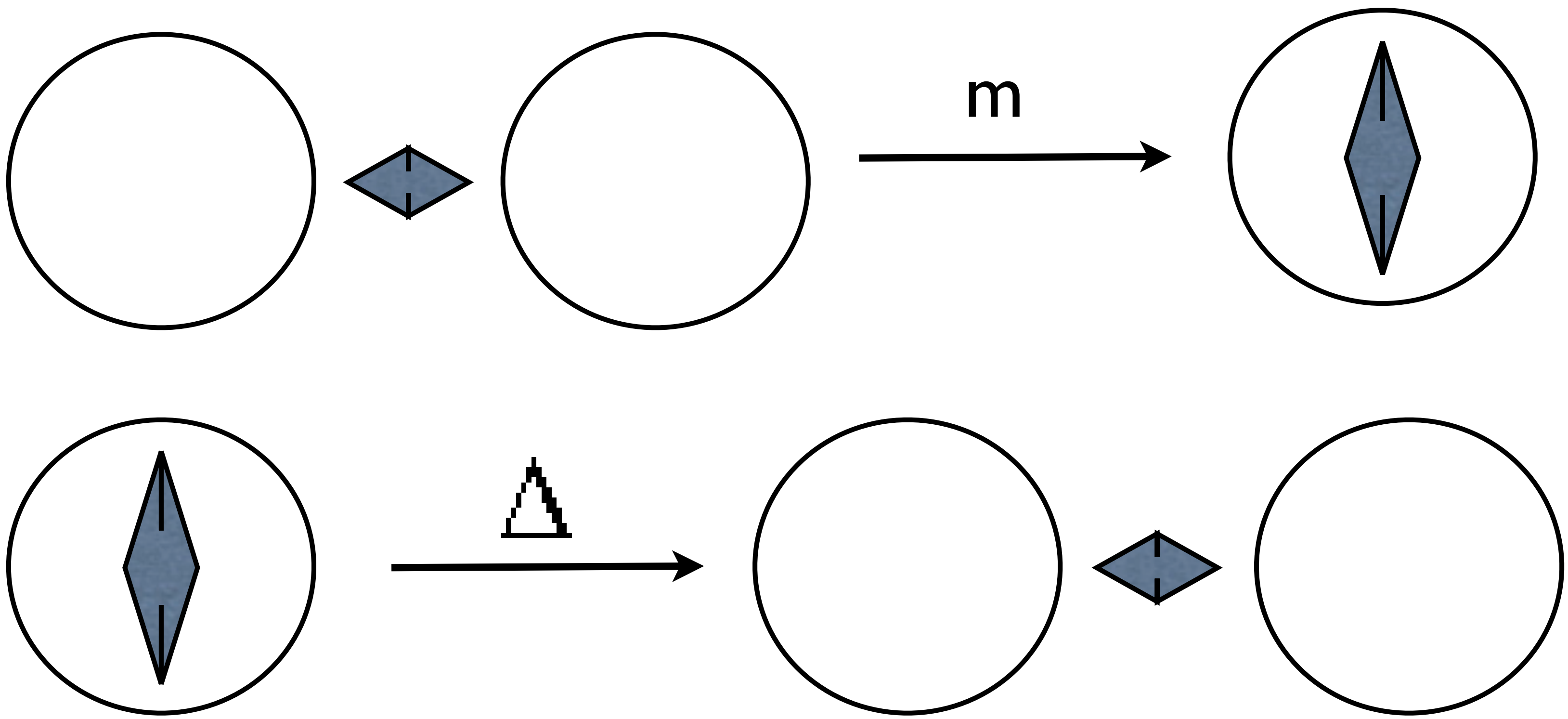


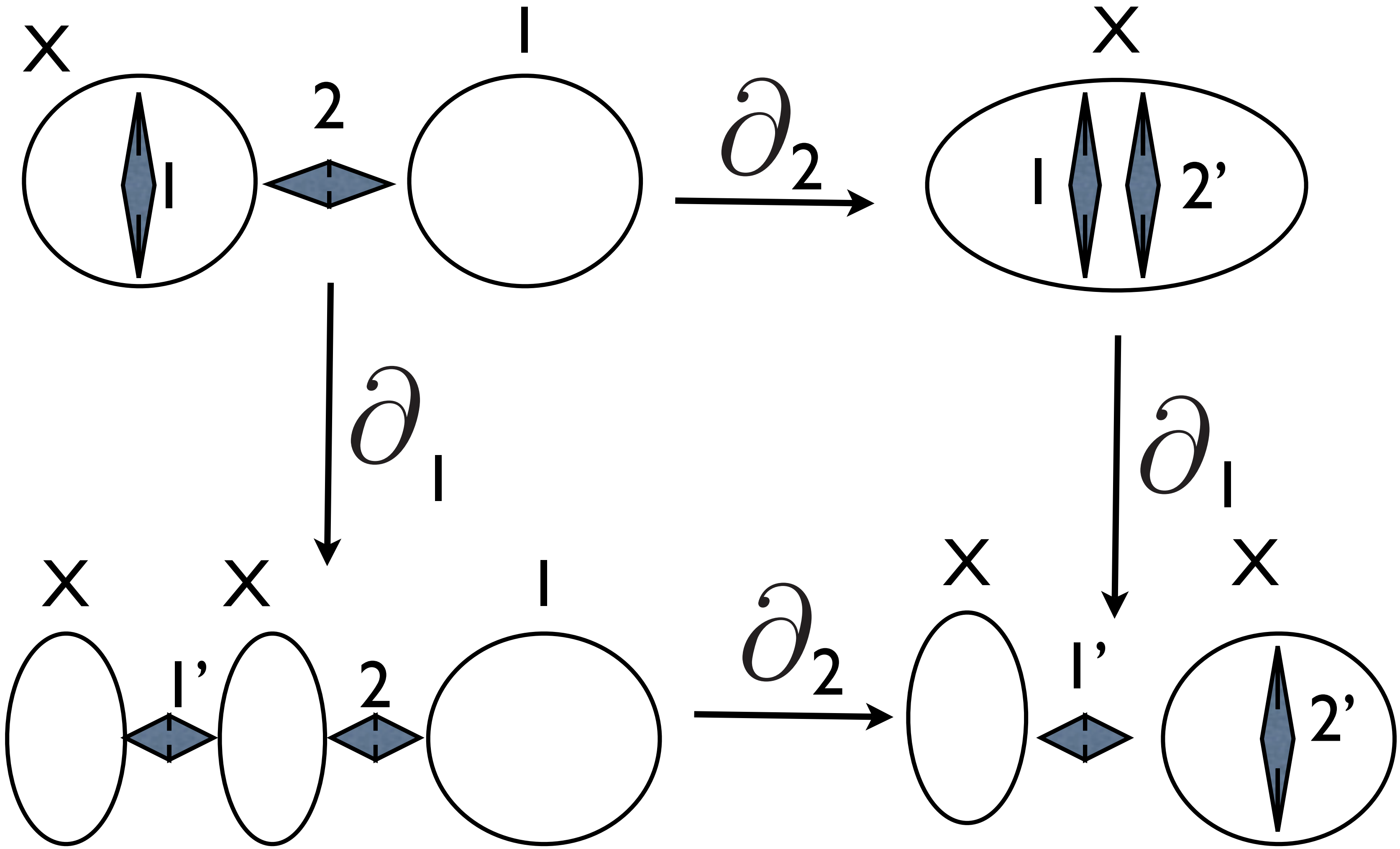
It turns out that one can take the algebra generated by 1 and X with

$X^2 = 0$ and

$$\Delta(X) = X \otimes X \text{ and } \Delta(1) = 1 \otimes X + X \otimes 1.$$

The chain complex is then generated by enhanced states with loop labels 1 and X.





An example of the commutation of partials.

Enhanced State Sum Formula for the Bracket

$$\langle K \rangle = \sum_s q^{j(s)} (-1)^{i(s)}$$

$$j(s) = n_B(s) + \lambda(s)$$

$i(s) = n_B(s)$ = number of B-smoothings in the state s.

$\lambda(s)$ = number of +1 loops minus number of -1 loops.

(X)

$$\langle K \rangle = \sum_{i\ ,j} (-1)^i q^j dim(\mathcal{C}^{ij})$$

C^{ij} = module generated by enhanced states with $i = n_B$ and j as above.

$$\langle K \rangle = \sum_{i\,,j} (-1)^i q^j dim(\mathcal{C}^{ij})$$

Khovanov constructs differential acting in the form

$$\partial : \mathcal{C}^{ij} \longrightarrow \mathcal{C}^{i+1\,j}$$

For j to be constant as i increases by 1, we need

$\lambda(s)$ to decrease by 1.

[go back two slides]

The differential increases the homological grading i by 1 and leaves fixed the quantum grading j.

Then

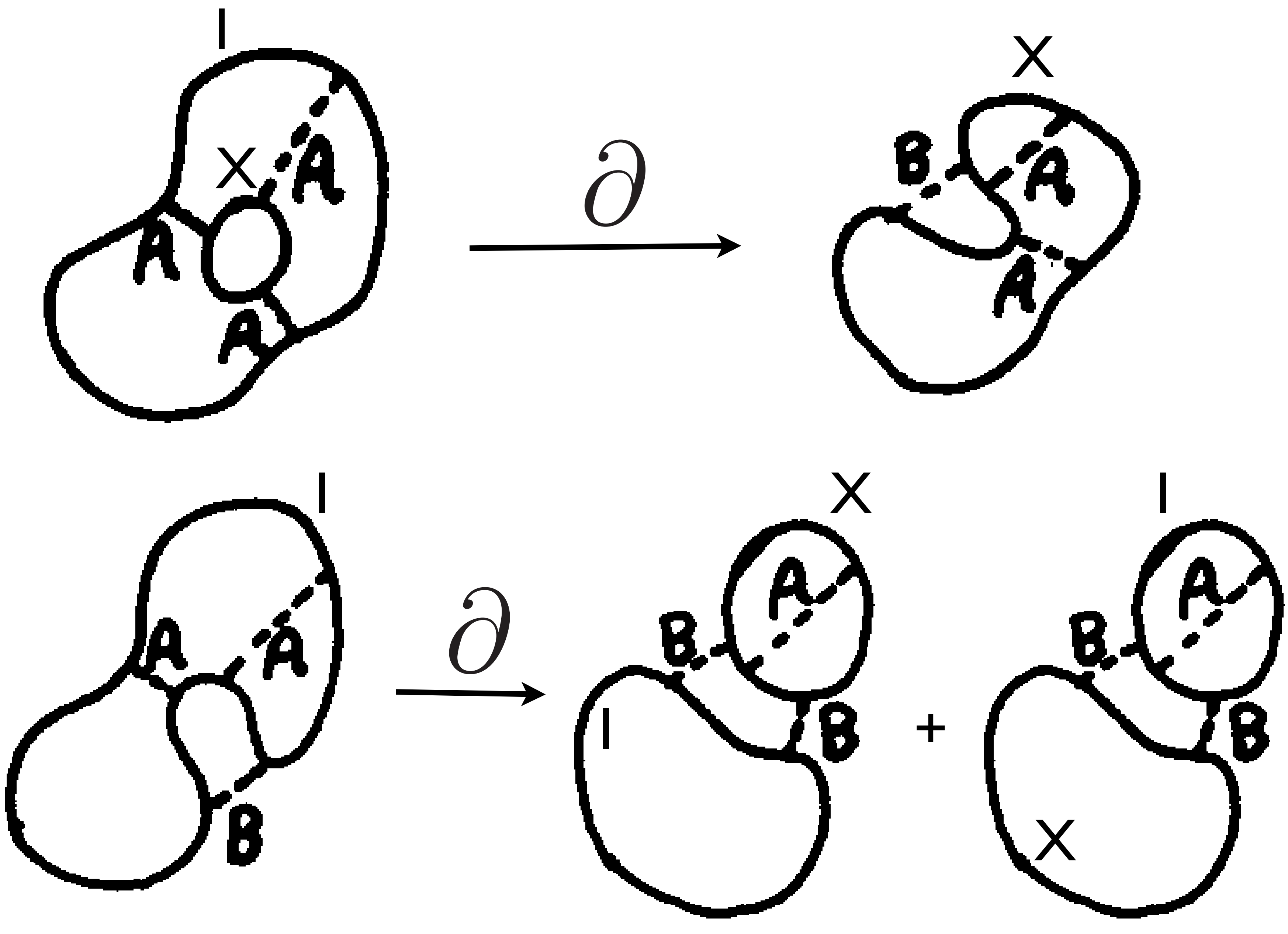
$$\langle K \rangle = \sum_j q^j \sum_i (-1)^i dim(\mathcal{C}^{ij}) = \sum_j q^j \chi(\mathcal{C}^{\bullet j})$$

$$\chi(H(\mathcal{C}^{\bullet j})) = \chi(\mathcal{C}^{\bullet j})$$

$$\langle K \rangle = \sum_j q^j \chi(H(\mathcal{C}^{\bullet j}))$$

$$\langle K \rangle = \sum_j q^j \chi(H(\mathcal{C}^{\bullet j}))$$

Partial Boundaries



A Quantum Statistical Model for Khovanov Homolgy and the Bracket Polynonmial.

Let C(K) denote a Hilbert space with basis |s> where s runs over the enhanced states of a knot or link diagram K.

We define a unitary transformation.

$$U : \mathcal{C}(K) \longrightarrow \mathcal{C}(K)$$

$$U|s\rangle = (-1)^{i(s)} q^{j(s)} |s\rangle$$

q is chosen on the unit circle in the complex plane.

C(K) = HilbertSpace(K)
is the direct sum
of the spaces V(S) where S ranges over the
orginal bracket states of the knot K.

Each V(S) is a tensor product of single qubit
spaces V.
Each single qubit space is endowed with a
Frobenius algebra structure.

$$|\psi\rangle = \sum_s |s\rangle$$

Lemma. The evaluation of the bracket polynomial is given by the following formula

$$\langle K \rangle = \langle \psi | U | \psi \rangle.$$

This gives a new quantum algorithm for the Jones polynomial (via Hadamard Test).

With $U|s\rangle = (-1)^{i(s)} q^{j(s)} |s\rangle,$

$$\partial : \mathcal{C}^{ij} \longrightarrow \mathcal{C}^{i+1\,j}$$

$$U\partial + \partial U = 0.$$

This means that the unitary transformation U acts on the homology so that

U: H(C(K) -----> H(C(K))

U: H(C(K) -----> H(C(K))

This means that the Khovanov Homology itself is a natural Hilbert space for the Jones polynomial.

$$\mathcal{C}^{\bullet,j} = \bigoplus_i \mathcal{C}^{i,j}$$

$$\langle K \rangle = \sum_s q^{j(s)}(-1)^{i(s)} = \sum_j q^j \sum_i (-1)^i dim(\mathcal{C}^{ij})$$

$$= \sum_j q^j \chi(\mathcal{C}^{\bullet,j}) = \sum_j q^j \chi(H(\mathcal{C}^{\bullet,j})).$$

This shows how <K> as a quantum amplitude contains information about the homology.

Eigenspace Picture

$$\mathcal{C}^0 = \oplus_\lambda \mathcal{C}^0_\lambda$$

$$\mathcal{C}^\bullet_\lambda : \mathcal{C}^0_\lambda \longrightarrow \mathcal{C}^1_{-\lambda} \longrightarrow \mathcal{C}^2_{+\lambda} \longrightarrow \cdots \mathcal{C}^n_{(-1)^n\lambda}$$

$$\mathcal{C} = \oplus_\lambda \mathcal{C}^\bullet_\lambda$$

$$\langle\psi|U|\psi\rangle = \sum_\lambda \lambda\chi(H(C^\bullet_\lambda))$$

SUMMARY

We have interpreted the bracket polynomial as a quantum amplitude by making a Hilbert space C(K) whose basis is the collection of enhanced states of the bracket.

This space C(K) is naturally intepreted as the chain space for the Khovanov homology associated with the bracket polynomial.

$$\langle K \rangle = \langle \psi | U | \psi \rangle.$$

The homology and the unitary transformation U speak to one another via the formula

$$U\partial + \partial U = 0.$$

making H(C(K)) a natural setting for the quantum information.

Questions

We have shown how Khovanov homology fits into the context of quantum information related to the Jones polynomial and how the polynomial is replaced in this context by a unitary transformation U on the Hilbert space of the model. This transformation U acts on the homology, and its eigenspaces give a natural decomposition of the homology that is related to the quantum amplitude corresponding to the Jones polynomial.

The states of the model are intensely combinatorial, related to the representation of the knot or link.

How can this formulation be used in quantum information theory and in statistical mechanics?!